Multistage Decision Processes (1977)

Texas Tech University (1977)

R.J. Marks II Class Notes

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EE 5327
MULTISTAGE
DECISION PROCESSES
10:30 MWF / C&ME 208

8/29/77 (MON)

EE5327 INFINITE DIMENSIONAL THEORY
OF OPTIMIZATION

FUNCTIONAL ANALYSIS $\Leftrightarrow$ GEOMETRIC INTERPRETATION
BOILS DOWN TO TWO THEOREMS
1. PROJECTION THEOREM
2. HAHN - BANACH THEOREM
ALSO, DUALITY
COURSE REQUIRES
PRECISION (MATHEMATICAL)

NOTES

FUNCTIONS

CONSIDER 2 SETS A & B.

• DEFN: CARTESIAN PRODUCT (×) OF A & B =

$$A \times B = \{ \underbrace{(a, b)}_{\text{ORDERED PAIR (2-tuple)}} \mid a \in A,\ b \in B \}$$

EX: $A = \{1, 2, 3, 79\}$

$B = \{2, 4\}$

$A \times B = \{ (1,2), (1,4), (2,2), (2,4), \text{etc} \}$

CAN EASILY EXTEND TO n-FOLD
CARTESIAN PRODUCT

• DEFN: RELATION BETWEEN A & B =
ANY SUBSET $F \subset A \times B$

SUBSET

(A COLLECTION OF 2 TUPLES)

$f : A \to B$

DOMAIN: THE DOMAIN OF f IS A

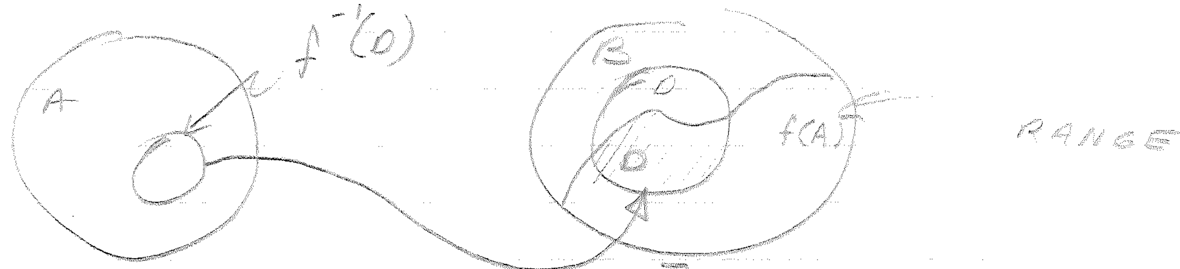
ARRIVAL SET: THE ARRIVAL SET OF f IS B

IMAGE OF A SUBSET $C \subset A$:
$f(C) = \{ b \mid b \in B, b = f(a) \text{ FOR SOME } a \in C \}$

RANGE (IMAGE) OF f: $f(A)$

INVERSE IMAGE OF A SUBSET $D \subset B$:
$f^{-1}(D) = \{ a \mid a \in A, f(a) \in D \}$

PICTURE



NOTE: $D \neq f[f^{-1}(D)]$ IN GENERAL.

H.W: READ CHAPT 1 & FIRST IN CHAPT 2.

NOTATION: $! \Leftrightarrow$ UNIQUE

9/7/77

TOPOLOGY

OPEN & CLOSED SETS

PROVE $\alpha\theta = \theta$

$(\alpha\beta)x = \alpha(\beta x)$

LET $\beta = 0$

USE "7" TO GET IT

SCALARS

<u>A FIELD</u> (10 AXIOMS) $\{\mathcal{F}, +, \cdot\}$

1. CLOSURE

$a, b \in \mathcal{F} \Rightarrow a + b \in \mathcal{F}$

2. $a + b = b + a$ (COMMUTATIVE)

3. ASSOC:

$a + (b + c) = (a + b) + c$

5 ADDITIVE INVERSE: $\forall a \in \mathcal{F} \ni -a \in \mathcal{F}$

$\ni a + (-a) = 0$

4. EXISTANCE & UNIQUENESS OF NEUTRAL ELEMENT

$\exists! 0 \in \mathcal{F} \ni \forall a \in \mathcal{F} \quad a + 0 = a$

6. CLOSURE $a, b \in \mathcal{F} \Rightarrow ab \in \mathcal{F}$

7. COMMUTATIVE, $ab = ba$

8. ASSOC, $a(bc) = (ab)c$

9. EXISTANCE & UNIQUENESS OF MULT. NEUTRAL EL.

$\exists! 1 \in \mathcal{F} \ni a \cdot 1 = a$

10. MULTIPLICATIVE INVERSE

$\forall a \neq 0 \in \mathcal{F} \ni a^{-1} \in \mathcal{F} \ni a a^{-1} = 1$

11. DISTRIBUTIVE

$a(b + c) = ab + ac$

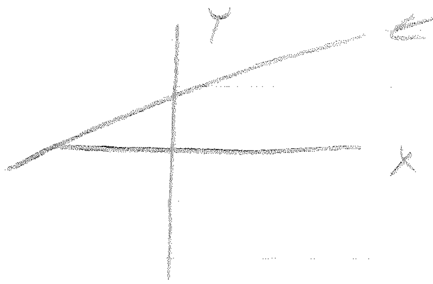
A PROP:

$b[a - a] = ab - ab = 0 \Rightarrow b \cdot 0 = 0$

TEST WED. MATERIAL THRU FRI

BACK TO PROBLEM

$$y = y_0 + Lx \quad (*)$$



← A LINEAR VARIETY!

SHOWING CONVEXITY

GIVEN (x_1, y_1) , (x_1, y_1),

SHOW

$\alpha(x_1,y_1) + (1-\alpha)(x_2,y_2)$ IS AN (x,y) PAIR SATISFYING $*$

$$y_1 = y_0 + Lx_1 \quad , \quad y_2 = y_0 + Lx_2$$

$$[\alpha x_1 + (1-\alpha)x_2 , \alpha y_1 + (1-\alpha) y_2]$$

$$y \overset{?}{=} y_0 + L[\alpha x_1 + (1-\alpha)x_2]$$

$$= y_0 + \alpha L x_1 + (1-\alpha) L x_2 + \alpha y_0 - \alpha y_0$$
$$= \alpha[y_0 + Lx_1] + (1-\alpha)[y_0 + Lx_2]$$
$$= \alpha y_1 + (1-\alpha) y_2$$

REF: "ELEMENTRY GENERAL TOPOLOGY"
HERAL
T.O. MOORE

9-17-77 (MON)

9-20-77 (WED)

p. 27, PROP 2:

A SET F IS CLOSED $\Leftrightarrow$ EVERY CONVERGENT SEQUENCE $\{x_n\}$, $x_n \in F$ HAS ITS LIMIT IN F

$\ni x \rightarrow x_n$

PROOF:

SUFFICIENCY ($\Longrightarrow$)

1. $\{x_n\} \ni x_n \in F$
2. THEN x IS A CLOSURE POINT OF $F \ni x_n \rightarrow x$
3. BUT F IS CLOSED

$\Rightarrow x \in F$

NECESSITY ($\Longleftarrow$)

1. $x_n \rightarrow x \Rightarrow x \in F \ni x_n \in F$
2. CONTRAPOSITIVE:

SUPPOSE F IS NOT CLOSED

$\Longrightarrow$ CLOSURE POINT $x \notin F$

3. CHOOSE SPHERE $S(x, \frac{1}{n})$. IN $S(x, \frac{1}{n})$ WE CAN ALWAYS SELECT SOME $x_n \in F \ \forall n$
4. $\{x_n\}$ CONVERGES TO $x \notin F$

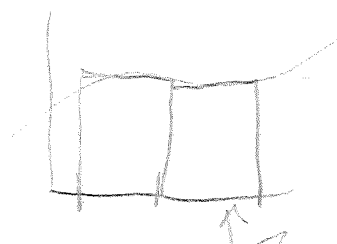
(A COUNTERICTION)

$\therefore$ F IS CLOSED.

9-23-77 (MON)

TEST #1 SCORES FROM 38's TO 98

RIEMAN



LEBESQUE

RIEMAN $\subseteq$ LEBESQUE

QUIZ ON FRI

9-26-77 (MON)

$x \sin 1/x$ } ONE IS BOUNDED
$x^2 \sin 1/x$ } VARIATION & THE OTHER AIN'T

- SEMINORM IS A MAPPING $\|\cdot\|_s : \mathfrak{X} \to R$
WITH THE PROPERTY

1. $\| \alpha x \|_s = |\alpha| \, \|x\|_s$
2. $\| x + y \|_s \leq \|x\|_s + \|y\|_s$
3. $\|x\|_s \geq 0$ [DO NOT HAVE $\|x\|_s = 0 \Leftrightarrow x = \theta$]

EX: $\mathfrak{X} = R^n$

DEFINE A SEMI-NORM:

$\|x\|_s \triangleq x^T Q x \ni Q \in$ POS. SEMI-DEF. MATRIX

[IE POS. DEF $\Rightarrow x^T Q x \geq 0$, $= 0$ IFF $x = 0$]

LET $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$x^T Q x = x_1^2$

$x = (0, x_2) \in R^2 \Longrightarrow \|x\|_s = 0$

9-29-77 (WED.)

PROP: $\bigcup \mathcal{C} = \phi$ & $\bigcap \mathcal{C} = X$ FOR $\mathcal{C}$ = EMTY COLLECTION OF SUBSETS OF X

($\bigcup \mathcal{C} = \phi$). LET $p \in X$. FOR $p \in \bigcup \mathcal{C}$ THERE MUST BE AT LEAST ONE $A \in \mathcal{C} \ni p \in A$. BUT $\exists$ NO A'S $\in \mathcal{C}$ $\Rightarrow p \notin \bigcup \mathcal{C}$ AND p WAS ARBITRARY IN $X \Rightarrow \bigcup \mathcal{C} = \phi$

($\bigcap \mathcal{C} = X$)

SUPPOSE $p \in X$. IN ORDER THAT $p \notin \bigcap \mathcal{C}$, THERE MUST BE AN $A \in \mathcal{C}$ WHICH DOES NOT CONTAIN p. BUT THERE ARE NO A'S IN $\mathcal{C} \Rightarrow p$ BELONGS TO EVERY A IN $\mathcal{C}$ & THUS BY DEF $\bigcap \mathcal{C} = X$.

10-3-77 (MON)

10-5-77 (WED)

Ricia $R^3 \Rightarrow \{x_1, x_2, x_3\}$ e_i

$$(x|Y) = \sum_{i=1}^{3} x_i Y_i$$

10-7-77 (FRI) TEST ON 19TH

10-10-77 (MON)

QUIZ: 20 POINTS A PEACE

HI: 86

1. $x = \sum_{i=1}^{n} \alpha_i a_i$

$\{x, a_1, a_2, \ldots, a_n\}$ L.F

$$\theta = \sum_{i=1}^{n} \beta_i a_i + \beta_0 x$$

$$x - \sum \alpha_i a_i = \theta$$

$\beta_i - \quad i = 1, 2, \ldots n, = \alpha_i$

FOR SOME K

SHOW $B = \{x, a_1, \ldots a_{k-1}, a_k, \ldots, a_k\}$

IS BASIS OF $\underline{X}$

$$x = \sum_{\substack{i=1 \\ i \neq k}}^{n} x_i a_i + x_k a_k$$

$$a_k = \frac{1}{x_k} x - \sum_{\substack{i=1 \\ i \neq k}}^{n} a_i x_i$$

$$\forall\ y \in \underline{X}, \quad y = \sum_{i=1}^{n} \beta_i a_i$$

$$\theta = \alpha_k x + \sum_{\substack{i=1 \\ i \neq k}}^{n} \alpha_i a_i$$

$$x = \sum_{\substack{i=1 \\ i \neq k}}^{n} x_i a_i + x_k a_k \quad ; x_k \neq 0$$

$$\Rightarrow \theta = \alpha_k x_k a_k + \sum_{\substack{i=1 \\ i \neq k}}^{n} (\alpha_i + \alpha_k x_i) a_i$$

$$\Rightarrow \alpha_k x_k = 0$$

$$\alpha_i + \alpha_k x_i = 0 \qquad i = 1, \ldots k-1, k+1, \ldots n$$

$$\Rightarrow \alpha_i = 0$$

3. $p(t) = p_0 + p_1 t + \ldots + p_n t^n$

PICK AS BASIS

$\beta = \{1, t, t^2, t^3, \ldots\}$

IS IT LINEARLY DEPENDENT?

TO CHECK L.I. OF B, FORM AN ARBITRARY LINEAR COMBINATION OF ELEMENTS FROM B AND SET IT TO Θ, ie, $p(t) = 0 \; \forall t$

$$0 = p_1 t^{\alpha_1} + p_2 t^{\alpha_2} + \ldots + p_n t^{\alpha_n}$$
$$\ni 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n \quad \forall t \; \& \; \forall n$$

$\overset{?}{\Rightarrow}$ $p_1, p_2, \ldots p_n$ ARE ZERO.

$$\begin{bmatrix} 1 & t_1^{\alpha_1} & t_1^{\alpha_2} & \cdots & t_1^{\alpha_n} \\ 1 & t_2^{\alpha_1} & t_2^{\alpha_2} & & t_2^{\alpha_n} \\ 1 & t_3^{\alpha_1} & \vdots & & \\ \vdots & & & & \\ 1 & t_n^{\alpha_1} & t_n^{\alpha_2} & & t_n^{\alpha_n} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

THIS IS TO HAIRY.

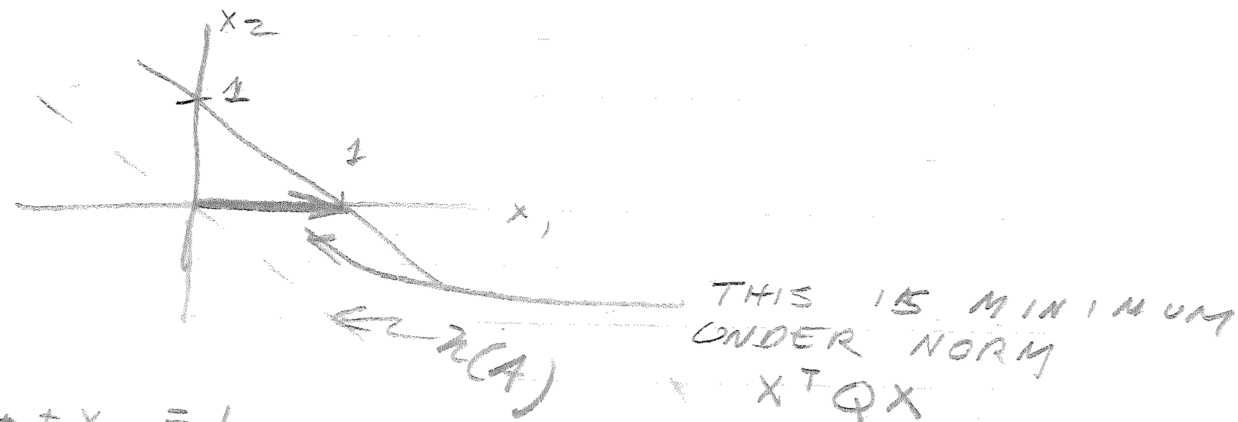
JUST DIFFERENTIATE, INSTEAD

$$\left(\frac{d}{dt}\right)^{\alpha_i} t^{\alpha_n} = \frac{\alpha_n!}{(\alpha_n - \alpha_i)!} \, t^{\alpha_n - \alpha_i}$$

10-12-77 (WED)
10-14-77 (FRI)
10-17-77 (MON)
Pg 76, #21

$\mathcal{N}(A) \equiv$ NULL SPACE OF A

EX: $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $A = [1 \ \ 1]$, $b = 1$



$x_1 + x_2 = 1$
$x_1 = 1 - x_2$

$$[x_1 \ \ x_2] \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ \ x_2] \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

$$= x_1^2 + 2x_1 x_2 + 2x_2^2$$
$$= (1-x_2)^2 + 2(1-x_2)x_2 + 2x_2^2$$
$$= 1 - 2x_2 + x_2^2 + 2x_2 - 2x_2^2 + 2x_2^2$$
$$= 1 + x_2^2 \Longrightarrow \text{MINIMUM FOR } x_2 = 0$$

10-24-77 (WED)

$\mathcal{L}: V \rightarrow W$

$\mathcal{L}[\alpha v_1 + \beta v_2] = \alpha \mathcal{L}[v_1] + \beta \mathcal{L}[v_2]$ ← LINEARITY

I'M INTERESTED IN FINITE-DIMENSIONAL V & W.

LET'S ASSUME THAT THERE ARE BASES SPECIFIED FOR V & W. LET V BE <u>n-DIMENSIONAL</u> & W BE m DIMENSIONAL. LET $\{v_1, v_2, \ldots, v_n\}$ BE THE BASIS FOR V & $\{w_1, w_2, \ldots, w_m)$ " " " " W. CONSIDER AN ARBITRARY $x \in V$. X CAN BE REPRESENTED IN THE BASIS FOR V. VIZUALIZE (VIZ).

$$x = \Sigma_{i=1}^{n} \xi_i v_i$$

CONSIDER

$\mathcal{L} v_i$. IT BELONGS TO W, HENCE

$$\mathcal{L} v_i = \Sigma_{j=1}^{m} \lambda_{ji} w_j$$

$$y = \mathcal{L} x = \Sigma_{i=1}^{n} \xi_i \mathcal{L}[v_i] = \sum_{i=1}^{n} \xi_i \sum_{j=1}^{m} \lambda_{ji} w_j$$

$$= \sum_{j=1}^{m} \Big[\underbrace{\sum_{i=1}^{n} \lambda_{ji} \xi_i}_{\text{COMPONENTS}}\Big] w_j$$

$$= \sum_{j=1}^{m} \eta_j w_j \Rightarrow \eta_j = \Sigma_{i=1}^{n} \lambda_{ji} \xi_i \quad j = 1, 2, \ldots, m$$

LET

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix} \qquad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix} \qquad \Rightarrow \eta = L \xi$$

$$L = \begin{bmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \lambda_{21} & & \\ \vdots & & \\ \lambda_{m1} & \cdots & \lambda_{mn} \end{bmatrix}$$

UBUS

10-31-77 (MON)

METRIC $d(x,y) \geq 0$

$d(x,y) = \|x-y\|$ (METRIC THOUGH, MAY NOT BE NORM)

LET $\mathcal{L}: C^n \to C^n$. LET $\{u_1, u_2, \ldots, u_n\}$ BE A BASIS FOR C^n & $L = \{\lambda_{ji}\}$ BE THE REPRESENTATION OF $\mathcal{L}$.

ie (THAT IS):

$$\mathcal{L} u_i = \sum \lambda_{ji} u_i \qquad (1)$$

LET $\{\hat{u}_1, \ldots, \hat{u}_n)$ BE A NEW BASIS OF C^n. LET'S FIND $\hat{L} = \{\hat{\lambda}_{ji}\}$ THAT REPRESENTS $\mathcal{L}$ IN THE NEW BASIS (IN TERMS OF L).

EXPAND NEW BASIS VECTORS IN TERMS OF THE OLD. ie

$$\hat{u}_i = \sum_{\alpha=1}^{n} C_{\alpha i} u_\alpha \qquad (2)$$

NOW CONSIDER AN ARBITRARY VECTOR $x \in C^n$. THE COMPONENTS OF x ARE DEFINED AS

$$x = \sum_{\alpha=1}^{n} \xi_\alpha u_\alpha = \sum_{\beta=1}^{n} \hat{\xi}_\beta \hat{u}_\beta .$$

FROM (2),

$$x = \sum_\beta \hat{\xi}_\beta \sum_\alpha C_{\alpha\beta} u_\alpha = \sum_\alpha \Big(\sum_\beta C_{\alpha\beta} \hat{\xi}_\beta\Big) u_\alpha$$

$$= \sum_{\alpha=1}^{n} \xi_\alpha u_\alpha$$

(3) $\Rightarrow$ $$\xi_\alpha = \sum_\beta C_{\alpha\beta} \hat{\xi}_\beta \quad ; \alpha = 1, 2, \ldots n$$

$$[\xi]_0 = C [\hat{\xi}]_0$$

IF L & $\hat{L}$ ARE THE MATRIX REPRESENTATIONS OF $\mathcal{L}$ W.R.T. THE CORRESPONDING BASIS.

$$\hat{L} = C^{-1} L C$$

IF $x \in \mathcal{C}$ AND IF $x = \sum_{k=1}^{n} \xi_k u_k = \sum_{i=1}^{n} \hat{\xi}_i \hat{u}_i$

THEN $\xi_k = \sum_{j=1}^{n} C_{kj} \hat{\xi}_j \quad ; k = 1, 2, \ldots, n$

RECALL FROM EQ. 2:

$$\hat{u}_i = \sum_{\alpha=1}^{n} C_{\alpha i} u_\alpha$$

IN COLUMN VECTOR NOTATION:

$$\vec{\hat{u}}_i = C_{1i}\begin{bmatrix} u_1 \end{bmatrix} + C_{2i}\begin{bmatrix} u_2 \end{bmatrix} + \cdots + C_{ni}\begin{bmatrix} u_n \end{bmatrix}$$

$$= [u_1 \ldots u_n]\begin{bmatrix} C_i \end{bmatrix} \Leftarrow i^{TH} \text{ COLUMN OF } C \text{ MATRIX}$$

$$\hat{U} \triangleq [\hat{u}_1 \; \hat{u}_2 \ldots \hat{u}_n]$$

$$= [u_1 \ldots u_n][C_1 \ldots C_n]$$

$$= UC$$

$$\Rightarrow \hat{U} = UC$$

$$U^{-1}\hat{U} = C \quad , \quad C^{-1} = \hat{U}^{-1} U$$

1140-27 (WED)

Pg 209 Pg. 1

$$f(x) = \max_{0 \le t \le 1} x(t)$$

$$\delta f(x;h) = \lim_{\alpha \to 0} \frac{1}{\alpha}\left[\max(x+\alpha h) - \max x\right]$$

LET t_0 BE $\ni$ $\max x = x(t_0)$

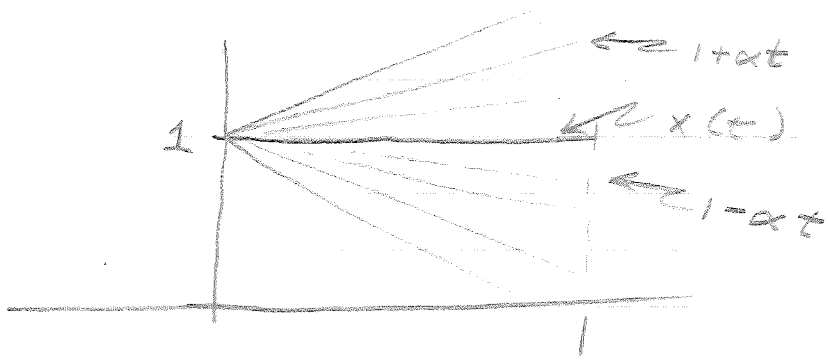
$\max\ x(t) + \alpha h(t) = x(t_0) + \alpha h(t_0)$

$\Rightarrow \delta f(x;h) = h(t_0)$ WHEN IT EXISTS

DOES NOT EXIST $\forall\ x$.

LET $x(t) = 1$, $h(t) = t$

$$\lim_{\alpha \to 0^+} \frac{1}{\alpha}\left[\max(1+\alpha t) - \max 1\right] \stackrel{?}{=} \lim_{\alpha \to 0^-} \frac{1}{\alpha}\left[\max(1+\alpha t) - \max 1\right]$$



$$\lim_{\alpha \to 0^+} \frac{1}{\alpha}\left[(1+\alpha) - 1\right]$$

$$\stackrel{?}{=} \frac{1}{\alpha}\left[(1 \quad - 1\right] \Rightarrow 1 \stackrel{?}{=} 0$$

Note: $x(t)$ ACHEIVES MAX @ MORE THAN ONE POINT

RIEMAN - STILTJES INTEGRAL

$$S_n = \sum_{i=1}^{n} f(\xi_i)\left[g(x_i) - g(x_{i-1})\right]$$

"BARTLE"
REAL ANALYSIS

$$\int_a^b f(x)\,dg(x) = \lim_{n\to\infty} S_n$$

$$= fg\Big|_a^b - \int_a^b g\,df$$

12-6-77 (LIBERTY'S LAST TEST LECTURE)

TAKE GATEAUX DIFFERENTIALS

$$\lim_{\alpha \to 0} \frac{1}{\alpha}\left[L(x+\alpha h, u+\alpha v) - L(x,u)\right]$$

GIVES

$$\delta L(x,u;h,v)$$

$$= \int_{t_0}^{t_f} [x^T Q h + u^T R v]\,dt$$

$$+ \int_{t_0}^{t_f} \lambda^T \left[h - \int_{t_0}^{t} A h\, d\tau - \int_{t_0}^{t} B v\, d\tau\right] dt$$

MUST SET $\delta L = 0 \ \forall (h,v)$

WANNA APPLY $\int (\cdot)\,dt = 0 \Rightarrow (\cdot) = 0$

$\forall$ DIFFERENTIABLE $(\cdot)$.

DEFINE

$$\rho(t) \triangleq -\int_{t}^{t_f} \lambda(\tau)\,d\tau$$

BY LEIBNITZ

$$\dot{\rho}(t) = \lambda(t)$$

THEN

$$\int_{t_0}^{t_f} \left[x^T(t) Q(t) h(t) + u^T(t) R(t) v(t)\right] dt$$

$$+ \int_{t_0}^{t_f} \dot{\rho}^T(t) h(t)\, dt$$

$$+ \int_{t_0}^{t_f} \lambda^T(t) \int_{t_0}^{t} A(\tau) h(\tau)\, d\tau$$

DERIV $= A(t)h(t)$

$$- \int_{t_0}^{t_f} \lambda^T(t) \int_{t_0}^{t} B(\tau) v(\tau)\, d\tau\, dt = 0$$

$$\rho^T(t_f) \int_{t_0}^{t_f} A(\tau) h(\tau)\, d\tau$$

$$- \rho^T(t_0) \int_{t_0}^{\cancel{t_0}} A(\tau) h(\tau)\, d\tau$$

INT. BY PARTS

$$- \int_{t_0}^{t_f} \rho^T(t) A(t) h(t)\, dt$$

NOW:

$$\dot{x} = Ax + Bu$$

$$u = R^{-1} B^T \rho$$

$$\Rightarrow \dot{x} = Ax - BR^{-1}B^T \rho \quad ; \quad x(t_0) = 0$$

$$\dot{\rho} = -A^T \rho - Qx \quad ; \quad \rho(t_f) = 0$$

$$\begin{bmatrix} \dot{x} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} A & -B^{-1}RB^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \rho \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x(t_0) \\ \rho(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ ? \end{bmatrix}, \quad \begin{bmatrix} x(t_f) \\ \rho(t_f) \end{bmatrix} = \begin{bmatrix} ? \\ 0 \end{bmatrix}$$

SOLUTION OF FORM

$$\begin{bmatrix} x(t) \\ \rho(t) \end{bmatrix} = \Phi(t, t_0) \begin{bmatrix} x_0 \\ \rho_0 \end{bmatrix}$$

ASSUME (REASONABLE FROM LINEARITY) =

$$\rho(t) = P(t)\, x(t)$$

$$\dot{\rho} = \dot{P}x + P\dot{x}$$
$$= \dot{P}x + PAx - PBR^{-1}B^T Px$$
$$= -A^T Px - Qx$$

$$\Rightarrow \left[\dot{P} - PA + A^T P - PBR^{-1}B^T P + Q\right] x = 0 \quad \forall x$$

$$\Rightarrow \dot{P} = -PA - A^T P - Q + PBR^{-1}B^T P$$

$$P(t_f) = 0$$

CAN SOLVE FOR P OFF LINE GIVEN SYSTEM.

EE 5327

"INFINITE DIMENSIONAL THEORY OF OPTIMIZATION"

- SOME SET THEORY

CONSIDER 2 SETS A & B

- DEFN: CARTESIAN PRODUCT ($\times$) OF A & B:

  $A \times B = \{(a,b) \mid a \in A, b \in B\}$

  NOTE: $A \times B \neq B \times A$ IN GENERAL

  CARTESIAN PRODUCT IS ORDERED 2-TUPLE

  CAN BE GENERALIZED TO n-FOLD CASE

- DEFN: RELATION BETWEEN A & B IS ANY SUBSET $F \in A \times B$.

  (A RELATION IS A SET OF 2-TUPLES)

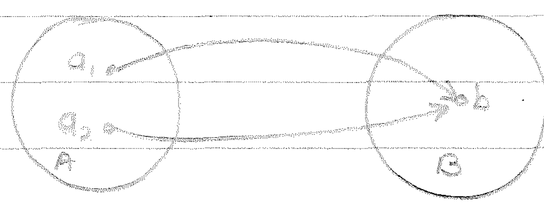
- NOTATION

  a IS RELATED TO b

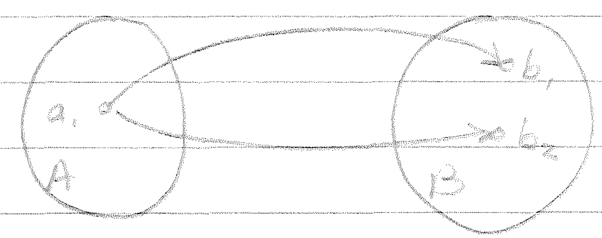
  $\equiv$ THE ORDERED PAIR $(a,b) \in F$

  $\equiv b = F(a)$

- FUNCTION: A FUNCTION FROM A TO B IS A RELATION BETWEEN A & B $\ni \forall a \in A \; \exists! \, b \in B \ni b = f(a)$



$\Leftarrow$ A FUNCTION



$\Leftarrow$ NOT A FUNCTION

- SYNONYMS:

  FUNCTION $\equiv$ MAPPING $\equiv$ MAP

  "f MAPS A INTO B" $\Rightarrow$ $f: A \to B$

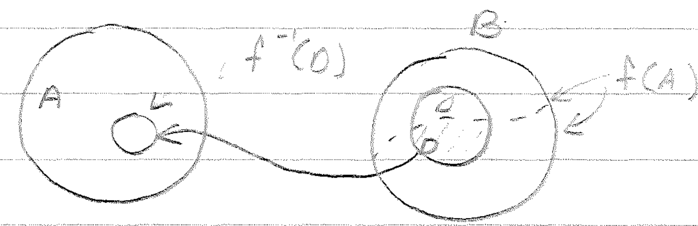
  NOTE: { ALL OF A MUST BE USED
  NOT ALL OF B NEED BE USED

- NOTE: A FUNCTION IS A SUBSET OF A CARTESIAN PRODUCT.
- DOMAIN: THE DOMAIN IN A MAP $f: A \to B$ IS A ["THE DOMAIN OF f IS A]
- ARRIVAL SET: THE ARRIVAL SET OF f IS B
- THE IMAGE OF A SUBSET $C \in A$:

  $$f[C] = \{ b \mid b \in B,\ b = f(a) \text{ FOR SOME } a \in C \}$$

- THE RANGE (IMAGE) OF f IS $f(A)$.
- INVERSE IMAGE OF A SUBSET: $D \subset B$

  $$f^{-1}(D) = \{ a \mid a \in A,\ f(a) = D \}$$



NOTE: IN GENERAL, $D \neq f[f^{-1}(D)]$

- A **1:1 FUNCTION** (ONE TO ONE)

  $f: A \to B$ IS 1:1 IF $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

- AN **ONTO** FUNCTION

  $f: A \to B$ IS ONTO IF $f(A) = B$

- **INVERSE** FUNCTION

  IF $f: A \to B$ IS 1:1, THE INVERSE

  $f^{-1}: f(A) \to A$

  IS DEFINED BY

  $$f^{-1}(b) = f^{-1}[f(a)] = a$$

- AXIOMS FOR A SCALAR FIELD $\{\mathfrak{F}, +, \cdot\}$

1. CLOSURE: $a, b \in \mathfrak{F} \Rightarrow a + b \in \mathfrak{F}$
2. " ; $a, b \in \mathfrak{F} \Rightarrow ab \in \mathfrak{F}$
3. ASSOCIATIVE: $a + (b + c) = (a + b) + c$
4. " ; $a(bc) = (ab)c$
5. EXISTANCE & UNIQUENESS OF A NEUTRAL ELEMENT

   $\exists !\, 0 \in \mathfrak{F} \ni \forall a \in \mathfrak{F}\;\; 0 + a = a$
6. COMMUTATIVE: $a + b = b + a$
7. " : $ab = ba$
8. ADDITIVE INVERSE: $\forall a \in \mathfrak{F} \;\exists\, -a \in \mathfrak{F} \ni a + (-a) = 0$
9. EXISTANCE & UNIQUENESS OF A MULTIPLICATIVE NEUTRAL ELEMENT: $\exists !\, 1 \in \mathfrak{F} \ni a \cdot 1 = a$
10. MULTIPLICATIVE INVERSE

    $\forall a \neq 0 \ni a \in \mathfrak{F} \;\exists\, a^{-1} \in \mathfrak{F} \ni aa^{-1} = 1$
11. DISTRIBUTIVE:

    $a(b + c) = ab + ac$

DEFINITION AND AXIOMS OF A VECTOR SPACE

$\underline{X} = \{ V, \mathcal{F}, +, \cdot \}$

$\alpha, \beta \in \mathcal{F}, \quad x, y \in V$

1. $x + y = y + x$ (COMMUTATIVE)
2. $(x + y) + z = x + (y + z)$ (ASSOCIATIVE)
3. $\exists\ \theta \in V \ni x + \theta = x \ \forall\ x \in V$
4. $\alpha(x + y) = \alpha x + \alpha y$
5. $(\alpha + \beta) x = \alpha x + \beta x$ } (DISTRIBUTIVE)
6. $(\alpha\beta) x = \alpha(\beta x)$ (ASSOCIATIVE
7. $0x = \theta \quad , \quad 1x = x$

PROPOSITIONS

1. $X + Y = X + Z \Rightarrow Y = Z$

PROOF: $-X = -X$

$$-X + (X + Y) = -X + (X + Z)$$

$$\Theta + Y = \Theta + Z \Rightarrow Y = Z$$

2. $\alpha X = \alpha Y \;\; \forall\, \alpha \neq 0 \Rightarrow X = Y$

PROOF: $\alpha^{-1} = \alpha^{-1} \Rightarrow \alpha^{-1}\cdot(\alpha x) = \alpha^{-1}(\alpha y) \Rightarrow x = y$

3. $\alpha X = \beta X \;\; \forall\, x \neq \Theta \Rightarrow \alpha = \beta$

PROOF: $x = \alpha^{-1}\beta X$

BUT $\alpha^{-1}\beta = 1$ IS UNIQUE

4. $(\alpha - \beta)x = \alpha X - \beta X$ (TRIVIAL)

5. $\alpha(x - Y) = \alpha X - \alpha Y$ (TRIVIAL)

6. $\alpha\Theta = \Theta$

PROOF $\alpha\Theta = \alpha(x - x)$

$$= \alpha x - \alpha x = \Theta$$

• PROP 2: LET M & N BE SUBSPACES OF X. THE $M+N$ IS A SUBSPACE OF X

PROOF: $M+N \ni \theta \Rightarrow M+N$ IS NON-EMPTY

$\forall\, m_1, m_2 \in M,\ n_1, n_2 \in N\ \exists$

$$x = m_1 + n_1,\quad y = m_2 + n_2$$

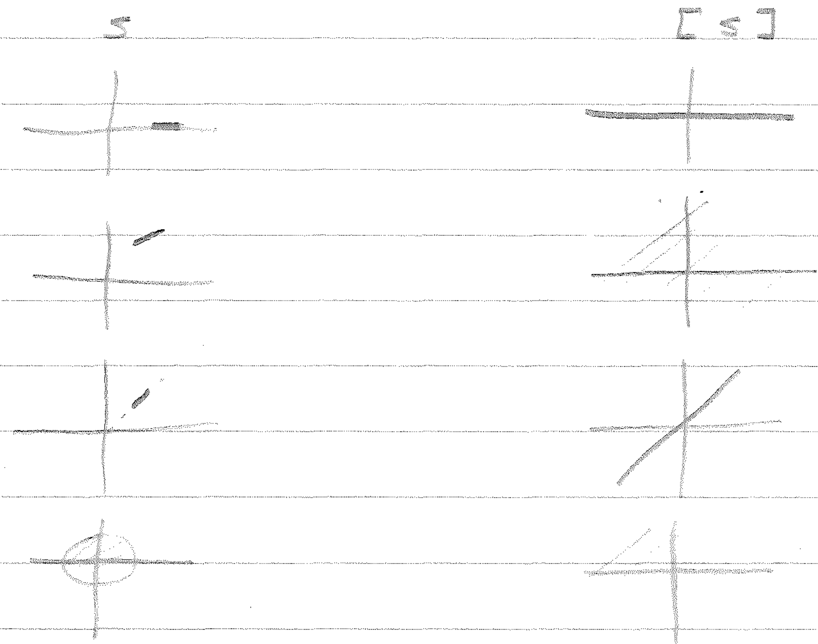
$$\therefore \forall\, \alpha, \beta \in \mathcal{F},\quad \alpha x + \beta y = \alpha(m_1+n_1) + \beta(m_2+n_2)$$
$$= (\alpha m_1 + \beta m_2) + (\alpha n_1 + \beta n_2)$$

AND $\alpha m_1 + \beta m_2 \in M$

$\alpha n_1 + \beta n_2 \in N$

- LINEAR COMBINATION OF $x_1, \ldots, x_2$ IN A VECTOR SPACE IS $\alpha_1 x + \alpha_2 x_2 + \ldots + \alpha_n x_n$
- $[S]$ IS THE SUBSPACE GENERATED BY S IF IT CONTAINS ALL VECTORS $\in X$ THAT ARE LINEAR COMBINATION

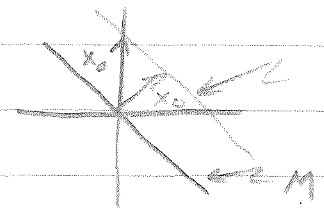
EXAMPLES (IN R^2):

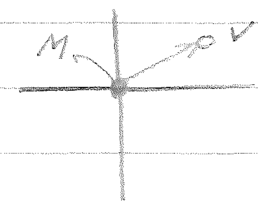


- A LINEAR VARIETY OF A SUBSPACE IS A TRANSLATION OF THAT SUBSPACE & CAN BE WRITTEN AS $V = x_0 + M$ ∋ M IS A SUBSPACE OF X. $x_0 \in X$

NOTE: FOR A GIVEN V, & M, x_0 IS NOT UNIQUE.

EXAMPLE





(HERE IT IS)

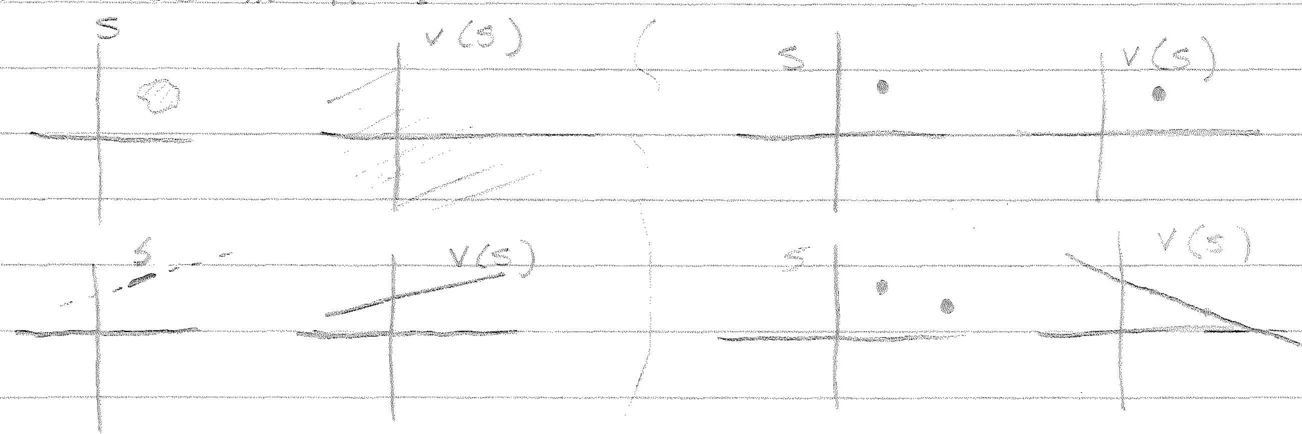
EQUIVALENT NAMES,

LINEAR VARIETY, FLAT, AFFINE SUBSPACE, LINEAR MANIFOLD.

SOME AUTHORS EQUATE "SUBSPACE" & "LINEAR MANIFOLD"

- LET S BE A NONEMPTY SUBSPACE OF A VECTOR SPACE X. THE LINEAR VARIETY GENERATED BY S, DENOTED $v(S)$, IS DEFINED AS THE INTERSECTION OF ALL LINEAR VARIETIES IN X THAT CONTAIN S.

EXAMPLES IN R^2:



PROVING A LINEAR VARIETY GENERATED BY S IS A LINEAR VARIETY. FIRST, IF TWO VARIETIES CONTAIN S, THEN THEIR INTERSECTION IS NOT EMPTY. FOR TWO SUCH LINEAR VARIETIES:

$$V_1 \cap V_2 = (M_1 + x_1) \cap (M_2 + x_2)$$
$$= (M_1 \cap M_2) + (x_1 \cap x_2)$$

FOR $x_1 \cap x_2$ TO BE NONEMPTY, $x_1 = x_2$. THUS

$$V_1 \cap V_2 = (M_1 \cap M_2) + x_1$$

$M_1 \cap M_2$ IS A SUBSPACE. THUS, $V_1 \cap V_2$ IS A LINEAR VARIETY.

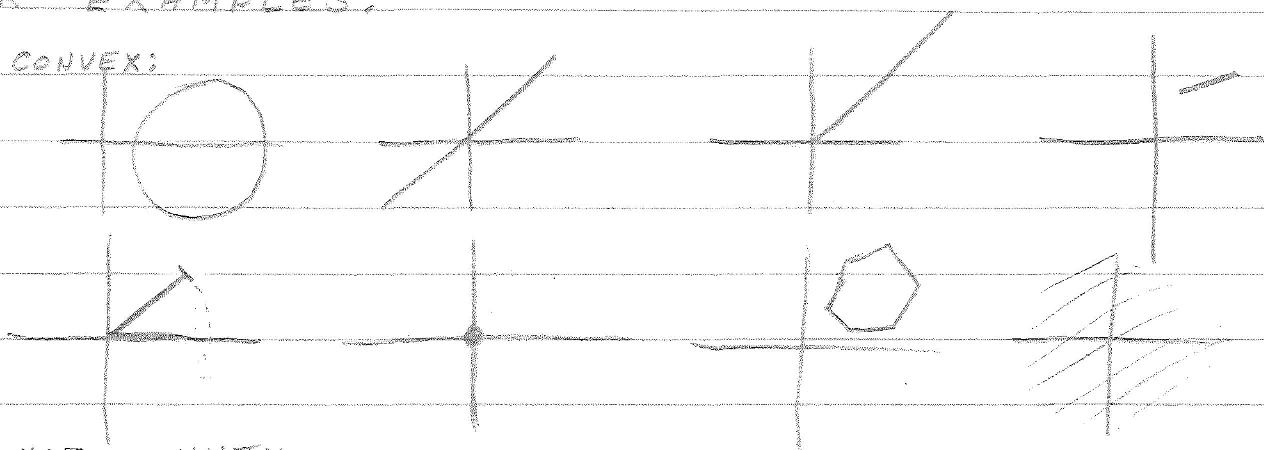
2-4 CONVEXITY & CONES

- A SET K IS A LINEAR VECTOR SPACE IS **CONVEX** IF $\forall\ x_1, x_2 \in K \Rightarrow 0 \le \alpha \le 1$

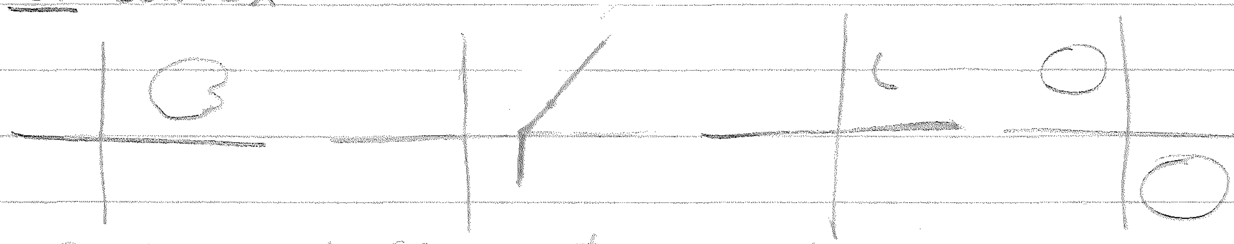
$$\Rightarrow \alpha x_1 + (1-\alpha) x_2 \in K$$

R^2 EXAMPLES:

CONVEX:



NOT CONVEX



NOTE: ALL SUBSPACES & LINEAR VARIETIES ARE CONVEX.

SUBSPACE: PROOF IS TRIVIAL

LINEAR VARIETY: $x_1, x_2 \in M + x_0 \ni M$ IS A SUBSPACE. $\therefore x_1 - x_0,\ x_2 - x_0 \in M$.

$$\therefore \quad \alpha(x_1 - x_0) + (1-\alpha)(x_2 - x_0)$$

$$\Rightarrow \alpha x_1 + (1-\alpha) x_2 \in M$$

$$\therefore \alpha x_1 + (1-\alpha) x_2 \in M + x_0$$

- PROP. 1: LET K & G BE CONVEX SETS IN A VECTOR SPACE. THEN

(a) $\beta K = \{x : x = \beta k, k \in K\}$ IS CONVEX

(b) $K + G$ IS CONVEX

PROOF: LET $\alpha \ni 0 \le \alpha \le 1$

(a) K IS CONVEX

$$\Rightarrow \alpha k_1 + (1-\alpha) k_2 \in K \quad \forall \; k_1, k_2 \in K$$

CONSIDER $\beta k \in \beta K$. THEN

$$\alpha[\beta k_1] + (1-\alpha)[\beta k_2]$$
$$= \beta[\alpha k_1 + (1-\alpha)\beta k_2] \in \beta K$$

(b) G IS CONVEX

$$\alpha g_1 + (1-\alpha) g_2 \in G$$

NOW, $k_1 + g_1 \in K + G$, $k_2 + g_2 \in K + G$.

$$\alpha[k_1 + g_1] + (1-\alpha)[k_2 + g_2]$$
$$= \underbrace{[\alpha k_1 + (1-\alpha) k_2]}_{\in K} + \underbrace{[\alpha g_1 + (1-\alpha) g_2]}_{\in G} \in K + G$$

- PROP 2: LET $\mathcal{C}$ BE AN ARBITRARY COLLECTION OF CONVEX SETS. THEN $\bigcap_{K \in \mathcal{C}} K$ IS CONVEX.

PROOF: LET $C = \bigcap_{K \in \mathcal{C}} K$. IF C IS EMPTY, THE THEOREM IS TRIVIALLY PROVED.

LET $x_1, x_2 \in C$ & $0 \leq \alpha \leq 1$. THEN $x_1, x_2 \in K$ $\forall K \in \mathcal{C}$. SINCE K IS CONVEX,

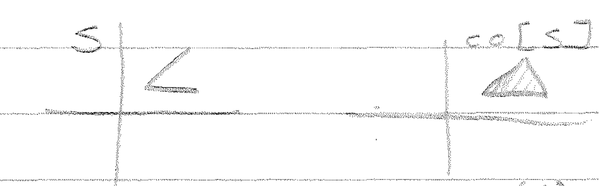
$\alpha x_1 + (1-\alpha) x_2 \in K \ \forall K \in \mathcal{C}$.

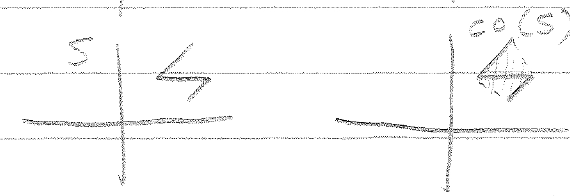
$\therefore \alpha x_1 + (1-\alpha) x_2 \in C \Rightarrow C$ IS CONVEX.

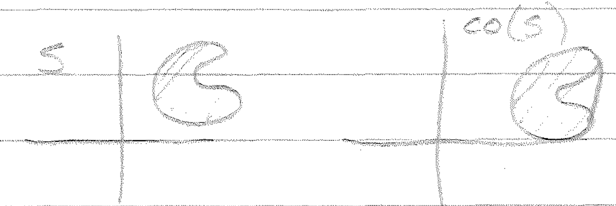
- LET S BE A SET IN A LINEAR VECTOR SPACE. THE CONVEX COVER OR CONVEX HULL, DENOTED $co(S)$, IS THE SMALLEST CONVEX SET CONTAINING S.

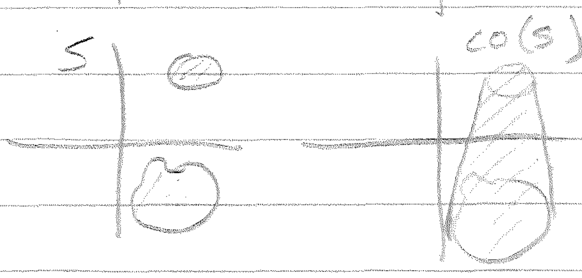
ie, $co(S)$ IS THE INTERSECTION OF ALL CONVEX SETS CONTAINING S.

R^2 EXAMPLES



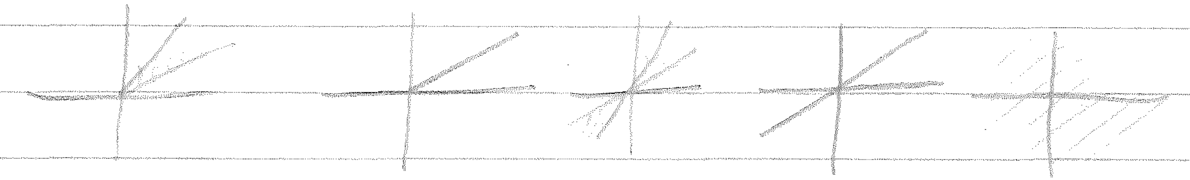




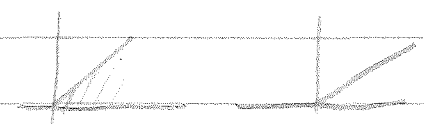


- A SET C IN A LINEAR VECTOR SPACE IS SAID TO BE A CONE WITH VERTEX @ THE ORIGIN IF $x \in C \Rightarrow \alpha x \in C \quad \forall \alpha > 0$.

EXAMPLES IN R^2:



- CONE WITH VERTEX @ $p = C + p$
- CONVEX CONE



2-5: LINEAR INDEPENDENCE & DIMENSION

- A VECTOR X IS SAID TO BE LINEARLY DEPENDENT ON A SET S OF VECTORS IF X CAN BE EXPRESSED IN A LINEAR COMBINATION OF VECTORS IN X.

  OR, EQUIVALENTLY,

  IF $x \in [S]$, i.e., $x = \sum_{i=1}^{n} \alpha_i y_i \ni y_i \in S$

- X IS LINEARLY INDEPENDENT OF S IF IT IS NOT LINEARLY DEPENDENT ON S.

• THM 1: A NEC. & SUFF CONDITION FOR VECTORS $x_1, x_2, \ldots, x_n$ TO BE INDEPENDENT IS THAT $\sum_{k=1}^{n} \alpha_k x_k = \theta \Rightarrow \alpha_k = 0 \ \forall k$.

PROOF:

NEC: ASSUME $\sum_{k=1}^{n} \alpha_k x_k = \theta$ & $\exists \alpha_r \neq 0$. THEN

$x_r = \sum_{k \neq r} \frac{-\alpha_k}{\alpha_r} x_k \Rightarrow x_r$ IS DEPENDENT ON THE OTHER x_k'S.

SUFF: ASSUME $x_r = \sum_{k \neq r} \alpha_k x_k$

$\Rightarrow \sum_{k \neq r} \alpha_k x_k - x_r = \theta$.

• CORR 1: IF x_i, $i = 1, \ldots, n$, THEN

$\sum_{k=1}^{n} \alpha_k x_k = \sum_{k=1}^{n} \beta_k x_k \Rightarrow \alpha_k = \beta_k$.

PROOF:

$\sum_{k=1}^{n} (\alpha_k - \beta_k) x_k = 0$

$\Rightarrow \alpha_k - \beta_k = 0$

- A SET S IS A **BASIS** FOR A VECTOR SPACE $\mathfrak{X}$ IF S GENERATES $\mathfrak{X}$. A SPACE HAVING A FINITE BASIS IS FINITE DIMENSIONAL. OTHERWISE THE SET IS INFINITELY DIMENSIONAL.

THEM 2: ANY TWO BASES FOR A FINITE-DIMENSIONAL VECTOR SPACE CONTAIN THE SAME # OF ELEMENTS.

PROOF: ASSUME

$\{x_1 \ldots x_n\}$, $\{Y_1 \ldots Y_m\}$, $m \geq n$ ARE TWO BASES

THEN

$$Y_j = \sum_{K=0}^{n} \alpha_{jK} X_K$$

$$X_j = \sum_{K=0}^{n} \beta_{jK} Y_j \Rightarrow n = m$$

2.6 NORMED LINEAR SPACES: DEF'S & EXAMPLE

- NORM: $\|\cdot\|$ IS A REAL-VALUED FUNCT'N WHICH MAPS EVERY $x \in \mathfrak{X}$ INTO $\|x\|$ & HAS THE FOLLOWING PROPERTIES:

  1. $\|x\| \geq 0 \ \forall x \in \mathfrak{X}$, $\|x\| = 0 \Leftrightarrow x = \theta$
  2. $\|x+y\| \leq \|x\| + \|y\| \ \forall x, y \in \mathfrak{X}$ (TRIANGLE INEQUALITY)
  3. $\|\alpha x\| = |\alpha| \cdot \|x\| \ \forall \alpha \in \mathfrak{F}, \forall x \in \mathfrak{X}$

- LEMMA: IN A NORMED LINEAR SPACE

$$\|x\| - \|y\| \leq \|x - y\| \quad \forall x, y \in \mathfrak{X}$$

PROOF: $\|x\| - \|y\| = \|x - y + y\| - \|y\|$

$$\leq \|x-y\| + \|y\| - \|y\| = \|x - y\|$$

2-7. OPEN & CLOSED SETS

- <u>INTERIOR POINT</u>: IF $P \in X$, $p \in P$, THEN p IS AN INTERIOR POINT IF $\exists\ \epsilon > 0 \ni$ ALL x SATISFYING $\|x - p\| < \epsilon$ ARE $\in P$. THE SET OF ALL INTERIOR POINTS OF P IS THE <u>INTERIOR</u> OF P AND IS DENOTED $\mathring{P}$.

  EX: $\overset{P}{(\longleftrightarrow]} \Rightarrow \overset{P^{\circ}}{(\longleftrightarrow)}$

- P IS <u>OPEN</u> IF $P = \mathring{P}$
- A POINT $x \in X$ IS A <u>CLOSURE POINT</u> (LIMIT POINT, CLUSTER POINT) OF P IF, FOR A GIVEN $\epsilon > 0\ \exists\ p \in P \ni \|x - p\| < \epsilon$. THE COLLECTION OF ALL CLOSURE POINTS OF P IS THE CLOSURE OF P & IS DENOTED BY $\bar{P}$.

  EX: $\overset{P}{(\longleftrightarrow]} \Rightarrow \overset{\bar{P}}{[\longleftrightarrow]}$

- P IS <u>CLOSED</u> IF $P = \bar{P}$

- PROP <u>1</u>: THE COMPLEMENT OF AN OPEN SET IS CLOSED AND A CLOSED SET IS OPEN.
- PROP 2: THE $\cap$ OF A FINITE # OF OPEN SETS IS OPEN. THE $\cup$ OF AN ARBITRARY COLLECTION OF OPEN SETS IS OPEN.

  NOTE: INTERSECTION MUST BE FINITE. DEFINE OPEN SET $(0, 1 + \frac{1}{n})$, $n = 1, 2, \ldots$

  THEN $\bigcap_{n=1}^{\infty} (0, 1 + \frac{1}{n}) = (0, 1]$

- PROP. 3: THE UNION OF A FINITE NUMBER OF CLOSED SETS IS CLOSED, T- INTERSECTION OF AN ARBITRARY COLLECTION OF CLOSED SETS IS CLOSED.
- PROP. 4: IF C IS CONVEX, SO IS $\bar{C}$ & $\mathring{C}$.
- LET V BE A LINEAR VARIETY. $p \in P$ IS AN INTERIOR POINT OF P RELATIVE TO V IF $\exists\ \varepsilon > 0 \ni$ EVERY $x \in V \ni \|x - p\| < \varepsilon$ ARE ALSO MEMBERS OF P.

2-8. CONVERGENCE:

- IN A NORMEL LINEAR SPACE (PRE-BANACH SPACE), A SEQUENCE $\{X_n\}$ CONVERGES IF $\{\|X - X_n\|\}$ OF REAL NUMBERS CONVERGES TO ZERO. WE DENOTE THIS BY $X_n \rightarrow X$

NOTE: $X_n \rightarrow X \Rightarrow \|X_n\| \rightarrow \|X\|$

- PROP 1: IF A SEQUENCE CONVERGES, IT'S LIMIT IS UNIQUE.
- PROP 2: A SET F IS CLOSED IFF EVERY CONVERGENT SEQUENCE WITH ELEMENTS IN F HAS ITS LIMIT IN F.

2-9: TRANSFORMATIONS & CONTINUITY

- LET $\mathbb{X}$ & $\mathbb{Y}$ BE LINEAR VECTOR SPACES, LET $D \subseteq \mathbb{X}$. A RULE WHICH ASSOCIATES WITH EVERY $x \in D$ AN ELEMENT $y \in \mathbb{Y}$ IS SAID TO BE A XFORMATION FROM $\mathbb{X}$ AND $\mathbb{Y}$ WITH DOMAIN D. $y = T(x)$
- A XFORMATION IS ONE-TO-ONE, $\exists \forall\, y \in \mathbb{Y}$ AT MOST ONE $x \in D \ni T(x) = y$
- IF $\forall\, y \in \mathbb{Y}$ $\exists$ AT LEAST ONE $x \in D$ $\ni T(x) = y$, T IS ONTO. i.e., WE MAP D ONTO $\mathbb{Y}$.
- AN XFORMATION FROM A VECTOR SPACE $\mathbb{X}$ INTO THE SET OF REAL (OR COMPLEX) NUMBERS, IS A FUNCTIONAL.

  EX. $f(x) = \|x\|$ IS A FUNCTIONAL
- T IS LINEAR IF $\forall\, \alpha_1, \alpha_2 \in \tilde{F}$ AND $\forall\, x_1, x_2 \in \mathbb{X}$, $T[\alpha_1 x_1 + \alpha_2 x_2] = \alpha_1 T(x_1) + \alpha_2 T(x_2)$
- LET T BE A MAPPING FROM NORMED SPACE $\mathbb{X}$ INTO A NORMED SPACE $\mathbb{Y}$.

  T IS CONTINUOUS AT $x_0 \in \mathbb{X}$ IF $\forall\, \varepsilon > 0\ \exists\ \delta > 0 \ni \|x - x_0\| < \delta \Rightarrow \|T(x) - T(x_0)\| < \varepsilon$

- PROP 1: T IS CONTINUOUS @ x_0 IFF $x_n \to x_0 \Rightarrow T(x_n) \to T(x_0)$

2-10. THE l_p AND L_p SPACES

- $p \in$ Pos. Int. l_p CONSISTS OF ALL SEQUENCES $\{\xi_1, \xi_2, \dots\} \ni$

$$\sum_{i=1}^{\infty} |\xi_i|^p < \infty \quad , \quad \|x\|_p = \left[\sum_{i=1}^{\infty} |\xi_i|^p\right]^{1/p}$$

l_∞ CONSISTS OF ALL BOUNDED SEQUENCES WITH NORM $\|x\|_\infty = \sup |\xi_i|$

- THE HÖLDER INEQUALITY

$p, q \in$ P.I. $\ni \frac{1}{p} + \frac{1}{q} = 1$, IF

$x = \{\xi_1, \dots\} \in l_p$ AND $y = \{\eta_1, \eta_2, \dots\}$

$$\Rightarrow \sum_{i=1}^{\infty} |\xi_i \eta_i| \le \|x\|_p \cdot \|y_q\|$$

SPECIAL CASES:

1. EQUALITY IFF $\left(\frac{|\xi_i|}{\|x\|_p}\right)^{1/q} = \left(\frac{|\eta_i|}{\|y\|_q}\right)^{1/p} \forall i$
2. CAUCHY SCHWARZ INEQUALITY: $p = q = 2$

$$\sum_{i=1}^{\infty} |\xi_i \eta_i| \le \left(\sum_{i=1}^{\infty} |\xi_i|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} |\eta_i|^2\right)^{1/2}$$

- THE MINKOWSKI INEQUALITY

$x, y \in l_p \Rightarrow x + y \in l_p$ AND $\|x+y\|_p \le \|x\|_p + \|y\|_p$

EQUALITY IFF $\exists k_1, k_2 > 0 \ni$

$k_1 x = k_2 y$

FOR CONTINUOUS FUNCTIONS:

- HÖLDER: $x \in L_p[a,b]$, $y \in L_q[a,b] \ni$

$1/p + 1/q = 1 \ni p, q > 1 \Rightarrow$

$$\int_a^b |x\, y|\, dt \le \|x\|_p \|y\|_q$$

- MINOWSKI: $x, y \in L_p[a,b]$

$\Rightarrow x + y \in L_p[a,b]$

AND $\|x+y\|_p \le \|x\|_p + \|y\|_p$

2·11: BANACH SPACES

- A SEQUENCE $\{x_n\}$ IN A NORMED SPACE IS SAID TO BE CAUCHY IF

$\|x_n - x_m\| \to 0$ AS $n, m \to \infty$. i.e.

$\forall \varepsilon > 0 \ \exists N \ni \|x_n - x_m\| < \varepsilon \ \forall \ n, m > N$

NOTE: IN A NORMED SPACE, ALL CONVERGENT SEQUENCES ARE CAUCHY. A CAUCHY SEQUENCE, HOWEVER, NEED NOT CONVERGE.

- A NORMED LINEAR VECTOR SPACE X IS COMPLETE IF EVERY CAUCHY SEQUENCE FROM X HAS A LIMIT IN X. THEN X IS A BANACH SPACE.

EX: LET X BE A SPACE OF CONTINUOUS FUNCTIONS ON $[0, 1]$ WITH NORM DEFINED BY $\|x\| = \int_0^1 |x(t)|\,dt$. X IS A NORMED LINEAR VECTOR SPACE, BUT IS <u>NOT</u> BANACH.

3 HILBERT SPACES

3.2 PRE-HILBERT SPACE (INNER PRODUCT)

$(x|y) = \overline{(y|x)}$

$(x+y|z) = (x|z) + (y|z)$

$\|x\|^2 = (x|x)$

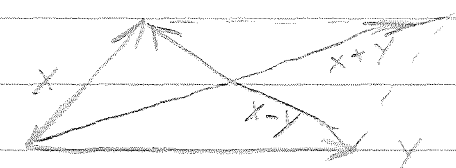
- CAUCHY-SCHWARZ INEQUALITY:

  $|(x|y)| \leq \|x\|\|y\|$

  EQUALITY IFF $x = \lambda y$ OR $y = \theta$

- $(x|y) = 0 \ \forall \ y \Rightarrow x = \theta$
- PARALLELOGRAM LAW (IN PRE-HILBERT SPACE)

  $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$



- HILBERT SPACE = COMPLETE PRE-HILBERT SPACE
- CONTINUITY OF INNER PRODUCT IN PRE-HILBERT SPACE

  $x_n \rightarrow x,\ y_n \rightarrow y \Rightarrow (x_n|y_n) \rightarrow (x|y)$

3.3 THE PROJECTION THEOREM

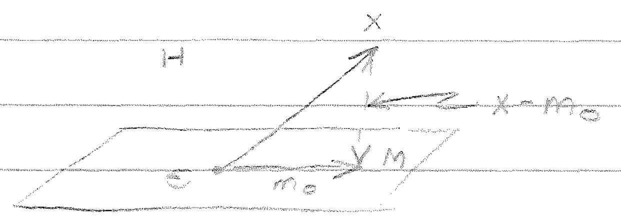
- ORTHOGONAL (IN PRE-HILBERT SPACE)

  $(x|y) = 0 \Rightarrow x \perp y$

- PYTHAGOREAN THEOREM: $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ IF $x \perp y$
- THE CLASSICAL PROJECTION THEOREM

  LET H BE A HILBERT SPACE & M A CLOSED SUBSPACE.

  $\forall x \in H \; \exists! \, m_0 \in M \ni \|x - m_0\| \leq \|x - m\| \; \forall \; m \in M$



$m_0 \in M$ MINIMIZING VECTOR $\Leftrightarrow (x - m_0 | M) = 0$
(ie, $x - m_0 \perp M$)

3.4. ORTHOGONAL COMPLEMENTS

- LET $S \subset H$. $S^{\perp}$ CONSISTS OF ALL VECTORS ORTHOGONAL TO S. $S^{\perp}$ = ORTHOGONAL COMPLEMENT.

PROPERTIES: S & T ARE SUBSETS OF A HILBERT SPACE

1. $S^{\perp}$ IS A CLOSED SUBSPACE $\forall\ S \subset H$
2. $S \subset S^{\perp\perp}$
3. $S \subset T \Rightarrow T^{\perp} \subset S^{\perp}$
4. $S^{\perp\perp\perp} = S^{\perp}$
5. $S^{\perp\perp} = \overline{[S]} \Rightarrow S^{\perp\perp}$ IS SMALLEST CLOSED SUBSPACE CONTAINING S.

- A VECTOR SPACE X IS A DIRECT SUM OF TWO SUBSPACES M & N IF $\forall\ x \in X\ \exists!\ m \in M, n \in N \ni x = m + n$. THIS IS DENOTED BY $X = N \oplus M$
- IF M IS A CLOSED LINEAR SUBSPACE OF H, THEN $H = M \oplus M^{\perp}$ AND $M = M^{\perp\perp}$

3-5. GRAM-SCHMIDT PROCEDURE

- A SET S IN PRE-HILBERT SPACE IS ORTHOG. SET IF $x \perp y \ \forall \ x, y \in S \ni x \neq y$. SET IS ORTHONORMAL IF $\|x\| = 1 \ \forall \ x \in S$.
- AN ORTHOGONAL SET OF NONZERO VECTORS IS LINEARLY INDEPENDENT SET.
- GRAM-SCHMIDT

  LET $\{x_i\}$ BE A LINEARLY IND. SET. THEN $\exists$ $\{e_i\}$ ORTHO. SET $\ni$ $[\{e_i\}] = [\{x_i\}]$

$$e_n = \frac{x_n}{\|x_n\|}$$

$$z_n = x_n - \sum_{i=1}^{n-1} (x_n | e_i) e_i$$

APPROXIMATION

3.6. THE NORMAL EQUATIONS & GRAM MATRICES

LET $y_i \in H$, $i=1,\dots,n$ & $M=[y_1 \dots y_n]$

LET $x \in H$. WE WISH TO FIND $\hat{x} \in M$ CLOSEST TO x. i.e., FOR $\hat{x} = \alpha_1 y_1 + \dots + \alpha_n y_n$, WE WANT TO MINIMIZE

$$\| x - \alpha_1 y_1 - \dots - \alpha_n y_n \| = \| x - \hat{x} \|$$

$\hat{x}$ IS ORTHOGONAL PROJECTION OF x ON M

$$\Rightarrow (x - \alpha_1 y_1 - \dots - \alpha_n y_n \mid y_i) = 0, \quad i=1,2,\dots,n$$

$$G = \text{GRAM MATRIX} = \begin{bmatrix} (y_1|y_1) & \cdots & (y_1|y_n) \\ & & \vdots \\ & & (y_n|y_n) \end{bmatrix}$$

$$g(y_1 \dots y_n) = \det[G]$$

$g \neq 0$ IFF $\{y_i\}$ ARE IND.

$$\delta = \| x - \hat{x} \| = \frac{g(y_1 \dots y_n, x)}{g(y_1 \dots y_n)}$$

3.7. FOURIER SERIES

- IF $S_n = \sum_{i=1}^{n} x_i$ CONVERGES TO x, THEN WE WRITE $\sum_{i=1}^{\infty} x_i = x$
- $\sum_{i=1}^{\infty} \xi_i e_i$ CONVERGES TO $x \in H$ IFF $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty \Rightarrow \xi_i = (x|e_i)$
- BESSEL'S INEQUALITY
  $\{e_i\}$ IS AN ORTHONORMAL SEQUENCE IN H.
  $$\Rightarrow \sum_{i=1}^{\infty} |(x|e_i)|^2 \leq \|x\|^2$$
- LET $S \subset H$. THEN THE "CLOSED SUBSPACE GENERATED BY S" IS $\overline{[S]}$
- LET $x \in H$. $\sum_{i=1}^{\infty} (x|e_i) e_i = \hat{x} \in M$ WHERE M IS THE CLOSED SUBSPACE GENERATED BY $\{e_i\}$.

3.8. COMPLETE ORTHONORMAL SEQUENCES

- $\{e_i\}$ IS COMPLETE IFF THE ONLY VECTOR ORTHOGONAL TO EACH e_i IS θ.

3.9. APPROXIMATION & FOURIER SERIES.

$\{y_i\}$ ARE IND.

FIND $\hat{x} \ni \|x - \hat{x}\|$ IS MIN

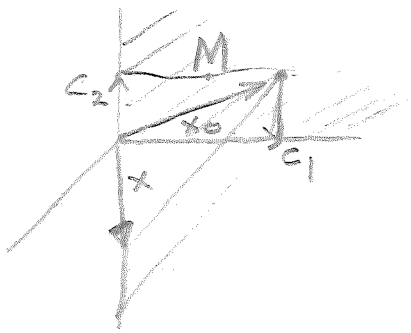
$\hat{x} = \sum_{i=1}^{n} (x \mid e_i) e_i$

APPLY GRAM SCHMIDT TO $\{y_1, y_2, \dots y_n, x\}$.
THEN $\|x - \hat{x}\|$ APPEARS IN FINAL STEP!

OTHER MINIMUM NORM PROBLEMS

3.10. THE DUAL APPROXIMATION PROBLEM

- LET $\{y_i\}$, $i = 1, 2, \ldots, n$ BE A SET OF IND. VECTORS IN H & $\{c_i\}$ CONSTANTS. CHOOSE ALL $x \ni (x|y_i) = c_i$. THE SET OF ALL x IS $M^\perp \ni M = [y_1 \ldots y_n]$



THEN THE x_0 WITH MINIMUM NORM IS

$$x_0 = \sum_{i=1}^{n} \beta_i y_i$$

WHERE β_i'S SATISFY

$$(y_1|y_1)\beta_1 + (y_2|y_1)\beta_2 + \ldots + (y_n|y_1)\beta_n = c_1$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$(y_1|y_n)\beta_1 + \qquad \ldots + (y_n|y_n)\beta_n = c_n$$

5. DUAL SPACES

5.2. BASIC CONCEPTS

LINEAR FUNCTIONAL: $f[\alpha x + y] = \alpha f(x) + f(y)$

FUNCTIONAL: A TRANSFORMATION FROM A VECTOR SPACE X INTO REAL #'S.

- IF A LINEAR FUNCTIONAL IS CONTINUOUS AT A POINT, THEN IT IS CONTINUOUS EVERYWHERE

D: A LINEAR FUNCTIONAL ON A NORMED SPACE IS BOUNDED IF $\exists M \ni |f(x)| \leq M\|x\| \;\forall x \in X$.

THE SMALLEST SUCH M IS THE NORM OF f:

$$\|f\| = \inf\{M : |f(x)| \leq M\|x\| \;\forall x \in X\}$$

- A LINEAR FUNCTIONAL ON A NORMED SPACE IS BOUNDED IFF IT IS LINEAR.

EX: AN UNBOUNDED LINEAR FUNCTIONAL

X = SPACE OF FINITELY NONZERO SEQUENCES

$x = \{\xi_1, \xi_2, \ldots \xi_n, 0, 0, \ldots\}$, $\|x\| = \max \xi_i$

$$f(x) = \sum_{k=1}^{n} k\xi_k$$

f IS OBVIOUSLY UNBOUNDED

- THE NORM OF A LINEAR FUNCTIONAL:

1. $\|f\| = \inf_M \{M : |f(x)| \leq M\|x\| \;\forall x \in X$

OR $M: \left|f\left(\frac{x}{\|x\|}\right)\right| \leq M$ GIVES ALTERNATE DEF:

2. $\|f\| = \sup_{x \neq 0} \left|f\left(\frac{x}{\|x\|}\right)\right|$

OR

$$\|f\| = \sup_{\|x\|=1} |f(x)|$$

OR

$$\|f\| = \sup_{\|x\|\leq 1} |f(x)|$$

EACH IS IDENTICAL

D: LET $\mathfrak{X}$ BE A NORMED LINEAR VECTOR SPACE. THE SPACE OF ALL BOUNDED LINEAR FUNCTIONALS ON $\mathfrak{X}$ IS THE NORMED <u>DUAL</u> OF $\mathfrak{X}$ AND IS DENOTED $\mathfrak{X}^*$. THE NORM OF AN ELEMENT $f \in \mathfrak{X}^*$ IS

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|$$

$x^* \in \mathfrak{X}$. THE VALUE OF x^* AT x IS

$$x^*(x) = \langle x, x^* \rangle.$$

- $\mathfrak{X}^*$ IS A BANACH SPACE

5.3. DUALS OF SOME COMMON BANACH SPACES

- THE DUAL OF E^n [n-DIMENSIONAL EUCLIDIAN SPACE]

  $x = (\xi_1, \xi_2, \ldots, \xi_n)$, $\|x\| = \sqrt{\sum^n \xi_i^2}$

  ALL LINEAR FUNCTIONALS ON E^n ARE OF THE FORM

  $$f(x) = \sum^n \xi_i \eta_i \; ; \; \|f\| = \left(\sum_{i=1}^n \eta_i^2\right)^{1/2}$$

  $f(x) = f\left[\sum \xi_i e_i\right] = \sum \xi_i f(e_i)$

  $\ni$ e_i's ARE BASIS VECTORS.

  ∴ THE DUAL OF E^n IS E^n

- THE DUAL OF l_p ; $1 \le p < \infty$

  $q = \frac{p}{p-1} \Rightarrow \frac{1}{p} + \frac{1}{q} = 1$

THEM: ALL BOUNDED LINEAR FUNCTIONALS ON l_p, $1 \le p < \infty$ CAN BE REPRESENTED AS

$$f(x) = \sum_{i=1}^{\infty} \eta_i \xi_i \;\ni\; y = \{\eta_i\} \in l_q$$

$$\|f\| = \|y\|_q = \begin{cases} \left(\sum_{i=1}^{\infty} |\eta_i|^q\right)^{1/q} & ; 1 < p < \infty \\ \sup_k |\eta_k| & ; p = 1 \end{cases}$$

∴ THE DUAL OF l_p IS l_q ; $p \neq \infty$

- THE DUAL OF L_p IS L_q
- THE DUAL OF c_0 [ALL INFINITE SEQUENCES $\{\xi_i\}$ OF REAL #'S CONVERGING TO 0.

  $\|x\| = \max_i |\xi_i| \Rightarrow c_0 \in l_\infty$

  THE DUAL OF c_0 IS l_1

- DUAL OF A HILBERT SPACE

  $f(x) = (x|y)$ FOR A FIXED y

  $\|f\| = \|y\|$

  IT TURNS OUT ALL LINEAR FUNCTIONALS ON A HILBERT SPACE ARE OF THE FORM $(x|y)$.

- RIESZ-FRÉCHET: IF f IS A BOUNDED LINEAR FUNCTIONAL ON A HILBERT SPACE H, THEN $\exists!\, y \in H \ni \forall x \in H$

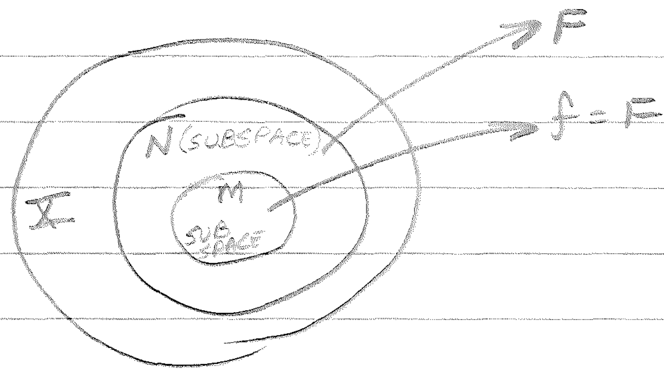
$$f(x) = (x \mid y)$$

ALSO, $\|f\| = \|y\|$ AND EVERY y DETERMINES A UNIQUE f.

EXTENSION FORM OF THE HAHN-BANACH THEOREM

5.4. EXTENSION OF LINEAR FUNCTIONALS

D: LET f BE A LINEAR FUNCTIONAL DEFINED ON A SUBSPACE M OF A VECTOR SPACE $\mathfrak{X}$. A LINEAR FUNCTIONAL F IS AN <u>EXTENSION</u> OF $f \ni F$ IS DEFINED ON A SUBSPACE N WHICH PROPERLY CONTAINS M



D: A REAL-VALUED FUNCTION p DEFINED ON REAL VECTOR SPACE $\mathfrak{X}$ IS A <u>SUBLINEAR FUNCTIONAL</u> IF

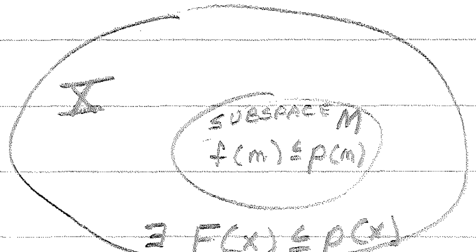
$$p(x_1 + x_2) \leq p(x_1) + p(x_2) \quad \forall\ x_1, x_2 \in \mathfrak{X}$$

$$p(\alpha x) = \alpha p(x) \quad \forall\ \alpha \geq 0,\ x \in \mathfrak{X}$$

A NORM IS A SUBLINEAR FUNCTIONAL

- HAHN-BANACH THEOREM: EXTENTION FORM

LET X BE A <u>REAL</u> LINEAR NORMED SPACE AND p A CONTINUOUS SUBLINEAR FUNCTIONAL ON X. LET f BE A LINEAR FUNCTIONAL DEFINED ON A SUBSPACE M OF $\mathfrak{X} \ni f(m) \leq p(m)\ \forall\ m \in M$. THEN $\exists$ AN EXTENSION F OF f FROM M TO $X \ni F(x) \leq p(x)$ ON $\mathfrak{X}$.



$$\|F\| = \|f\|_m$$

$$= \sup \frac{|f(m)|}{\|m\|}$$

- COROLLARY: LET f BE A BOUNDED LINEAR FUNCTIONAL DEFINED ON A SUBSPACE M OF A REAL NORMED VECTOR SPACE $\mathfrak{X}$. THEN $\exists$ A BOUNDED LINEAR FUNCTIONAL F DEFINED ON $\mathfrak{X}$ WHICH IS AN EXTENTION OF f WHICH HAS NORM EQUAL TO THE NORM OF f ON M.
- COROLLARY: LET $x \in \mathfrak{X}$ = NORMED SPACE. THEN $\exists$ $F \neq 0$ ON $\mathfrak{X} \ni F(x) = \|F\| \|x\|$.

5.5. THE DUAL OF $C[a,b]$

- RIESZ REPRESENTATION THEOREM:

LET f BE A BOUNDED LINEAR FUNCTIONAL ON $X = C[a,b]$. THEN $\exists$ A FUNCTION v OF BOUNDED VARIATION ON $[a,b] \ni \forall x \in X$

$$f(x) = \int_a^b x(t)\,dv(t)$$

$\ni \|f\| = T.V.(v)$ ON $[a,b]$.

D: THE NORMALIZED SPACE OF FUNCTIONS OF BOUNDED VARIATION DENOTED $NBV[a,b]$ CONSISTS OF ALL FUNCTIONS OF BOUNDED VARIATION ON $[a,b]$ WHICH ARE CONTINUOUS FROM THE RIGHT ON $[a,b]$ AND VANISH @ a.

$$\|v\| = T.V.(v)$$

5.6. THE SECOND DUAL SPACE

- $x^* \in X^* \Rightarrow x^*(x) = \langle x, x^* \rangle$ IF VALUE OF x^* @ $x \in X$.

  NOW, $f(x^*) = \langle x, x^* \rangle$ DEFINES A FUNCTIONAL ON X^* WHICH IS LINEAR

  $|f(x^*)| = |\langle x, x^* \rangle| \leq \|x\| \|x^*\|$

  IN FACT, $\|f\| = \|x\|$

- THE SPACE OF ALL LINEAR FUNCTIONALS ON X^* IS X^{**} AND IS THE SECOND DUAL OF X. $\phi : X \to X^{**}$ ie $x^{**} = \phi(x)$

  $\|\phi(x)\| = \|x\|$

- A NORMED SPACE X IS REFLEXIVE IF $\phi : X \to X^{**}$ IS ONTO $\Rightarrow X = X^{**}$.

EX. $\ell_p \quad 1 < p < \infty$ IS REFLEXIVE.

$\ell_p^* = \ell_q \Rightarrow \ell_p^{**} = \ell_q^* = \ell_p$

EX. ℓ_1 IS NOT REFLEXIVE;

$\ell_1^* = \ell_\infty$ BUT $\ell_\infty^* \neq \ell_1$

EX. ALL HILBERT SPACES ARE REFLEXIVE

$X^* = X \Rightarrow X^{**} = X^* = X$

5.7. ALIGNMENT & ORTHOGONAL COMPLEMENTS

D. $x^* \in X^*$ IS ALIGNED WITH $x \in X$ IF

$$\langle x, x^* \rangle = \|x^*\|\|x\|$$

EX: $X = L_p[a,b]$, $X^* = L_q[a,b]$. x & y ALIGNED IF

$$\int_a^b x y \, dt = \left[\int |x|^p dt\right]^{1/p} \left[\int |y|^q dt\right]^{1/q}$$

ie, IF HÖLDER $\neq$ IS AN $=$.

EX: $X = C[a,b]$, Γ = SET OF POINTS $\ni |x(t)| = \|x\|$.

LET $x^*(x) = \int_a^b x(t)\, dv(t)$. x^* IS ALLIGNED WITH x IFF v VARIES ONLY ON Γ . . . (etc).

D: $x \in X$ AND $x^* \in X^*$ ARE ORTHOGONAL IF

$$\langle x, x^* \rangle = 0$$

IN HILBERT SPACE: $\langle x, x^* \rangle = (x \mid y)$

D: LET $S \subseteq X$. THEN $S^\perp$ = <u>ORTHOGONAL COMPLEMENT</u> OF S, CONSISTS OF ALL $x^* \in X^*$ ORTHOGONAL TO EVERY VECTOR IN S.

T: LET M BE A CLOSED SUBSPACE OF A NORMED SPACE X. THEN

$$^\perp M^\perp = M$$

EE 5327 Quiz 1 Fall 1977

Instructions: Open book (Luenberger) and class notes only. All papers must be handed in by 12:30. Make sure you show all reasoning. No yes or no answers only.

I. Let $A = \{1, 3, 5\}$
$B = \{dog, cat, mouse\}$

Let $f_1 = \{(1, dog), (1, cat), (3, cat), (3, mouse)\}$

$f_2 = \{(1, dog), (3, cat), (5, mouse)\}$

$f_3 = \{(1, dog), (3, dog), (5, dog)\}$

Which of the above relations are functions? Explain your answer.

II. Let A be the real line interval $[-1, 1]$.
Let B be the real line.

Which of the following relations are functions? Explain your answer.

a. $\sin^{-1} x$ b. $|x|$ c. $\cos x$

$$\frac{d^2x}{dt^2} = -a\,x$$
$$\Rightarrow x = A \sin at + B \cos Bt$$

III. Consider the differential equation

$$\frac{d^2}{dt^2} x(t) + 3 \frac{d}{dt} x(t) + 2\,x(t) = 0 .$$

(a) Demonstrate a basis for the vector space of all solutions to this equation.

(b) What is the dimension of this vector space?

IV. Let L be a mapping from a vector space $\overline{X}$ to a vector space $\overline{Y}$. That is, $L : \overline{X} \to \overline{Y}$.

L is said to be linear if

$$L(\alpha x_1 + \beta x_2) = \alpha L(x_1) + \beta L(x_2),$$

for any scalars α and β.

(a) Show that the set of all pairs (x, y) in $X \times Y$ that satisfy $y = y_0 + L(x)$ is convex.

(b) Is this set a linear variety? Prove or disprove it!

V. Let K and G be convex sets in a vector space. Show that $K + G$ is convex.

I.	10
II	10
III	10
IVa	9
IVb	10
V	10

98%

I. f_1 is not a function

(1, dog) (1, cat) $\rightarrow$ 2 elements assigned to 1

f_2 is a function

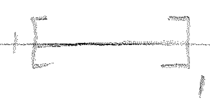
$\forall\, a \in A\ \exists$ a unique $b \in B \ni b = f_2(a)$

f_3 is a function

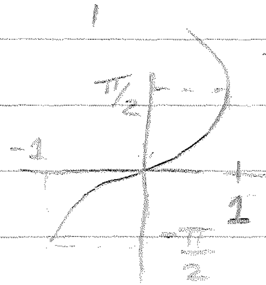
$\forall\, a \in A\ \exists$ a unique $b \in B \ni b = f_2(a)$

(In this case all b: are dogs.)

II. A B Assume $A \rightarrow B$



(a)



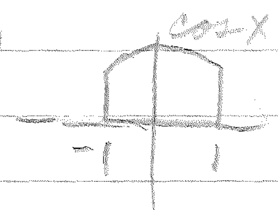
$\sin^{-1} x$ is not a function.

Let $x = 0 \Rightarrow b$

$b = \sin^{-1} x = 0, \pm\pi, \pm 2\pi, \ldots$

$\therefore$ b is not unique

(c)



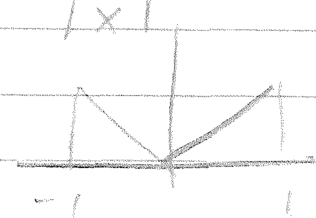
$\cos x$ is a function since $\forall\, a \in A$

$\exists$ a unique b in $B \ni b = \cos a$

[here b ranges from

$\cos 1$ to 1]

(b) $|x|$



$|x|$ is a function since

~~$\forall b \in b$~~ $\forall a \in A$ $\exists$ a unique

$b \in B \ni b = |A|$

III. $\frac{d^2}{dt^2}x(t) + 3\frac{d}{dt}x(t) + 2x(t) = 0$

~~$x(t) = Ae^{-t}$ is solution~~

also ~~$x(t) = Be^{-at}$~~

$a^2 - 3a + 2 = 0$

$(a-1)(a-2)$

$\Rightarrow x(t) = Ae^{-t} + Be^{-2t}$

vectors:

$V = \{ v \mid v = Ae^{-t} + Be^{-2t} \}$

$\ni A, B \in$ complex #'s

A Basis is

$\phi_1 = e^{-t}$

$\phi_2 = e^{-2t}$

$\forall v, \exists A, B \in$ complex #'s $\ni v = A\phi_1 + B\phi_2$

(b) The vector space is two dimensional

IV. $L : \mathbb{X}, \mathbb{Y}$

$\hat{x} \in \mathbb{X}$

$\hat{y} \in \mathbb{Y}$

$(\hat{x}, \hat{y}) \in \mathbb{X} \times \mathbb{Y}$

LET C be that subset of $\mathbb{X} \times \mathbb{Y} \ni$

$(x, y) \in C$, $y = L(x) + y_0$

$\therefore \forall x \in \mathbb{X}, \exists y_0 \ni (x, L(x) + y_0) \in C$

Choose $\gamma \ni 0 \leq \gamma \leq 1$

Then $\forall (x_1, y_1); (x_2, y_2) \in C$

$(\gamma x_1 + (1-\gamma) x_2, \gamma y_1 + (1-\gamma) y_2)$

NO

y_0's cancel

$= (\gamma x + (1-\gamma)x_2, \gamma L(x_1) + (1-\gamma) L(x_2))$

~~Now~~

$\cancel{L[\gamma x_1 + (1-\gamma)x_2]}$

$\cancel{= \gamma L(x_1) + (1-\gamma) L(x_2)}$

~~Since~~ $\gamma x_1 + (1-\gamma)x_2 \in \mathbb{X}$, ~~we conclude~~

$= (\gamma x_1 + (1-\gamma)x_2, L[\gamma x_1 + (1-\gamma) x_2] + y_0)$

Since $\gamma x_1 + (1-\gamma)x_2 \in \mathbb{X}$ then this $\in C$ and C is convex.

IV b. $L : X \to Y$

let $x \in X$. THEN

$(x, L(x) + y_0) \in C$

Is C a linear variety?

If so, $\exists\ x_1 \in X,\ y_1 \in Y \ni$

$\hat{C} = (x - x_1,\ L(x) + y_0 - y_1) = C - (x_1, y_1)$ is a subspace

Choose $x_1 = \theta,\ y_1 = y_0$ to give

$\hat{C} = \{ c \mid c = (x, L(x)) \}$

On pg. 12 of Luenburger we obviously satisfy axioms 1 & 2.

For 3: $L(\theta) = \theta$ is okay.

By linearity of L axioms 4, 5, & 6 are also satisfied.

$\therefore\ \hat{C}$ is a subspace &

C is a linear variety.

V) We Choose $\alpha \ni 0 \le \alpha \le 1$

K is convex

$\Rightarrow \forall k_1 k_2 \in K$:

$\alpha k_1 + (1-\alpha) k_2 \in K$

also, $\forall\ g_1, g_2 \in G$

$\alpha g_1 + (1-\alpha) g_2 \in G$

Now, if :

$(k_1 + g_1),\ (k_2 + g_2) \in K + G$

$\forall\ k_1, k_2 \in K,\ g_1, g_2 \in G$

$\alpha[k_1 + g_1] + (1-\alpha)(k_2 + g_2)$

$$= \underbrace{[\alpha k_1 + (1-\alpha) k_2]}_{\in K} + \underbrace{[\alpha g_1 + (1-\alpha) g_2]}_{\in G}$$

$\therefore\ \alpha[k_1 + g_1] + (1-\alpha)(k_2 + g_2) \in K + G$

$\Rightarrow K + G$ IS CONVEX.

EE 5327 Quiz 2

1. Let $\mathcal{A} = \{a_1, a_2, a_3, \dots, a_n\}$ constitute a basis of $\mathcal{X}$ (a vector space). If $x \in \mathcal{X}$, with $x \neq \theta$ show that $\{x, a_1, a_2, \dots, a_n\}$ is a linearly dependent set of vectors and that for some k, $1 \leq k \leq n$, the set $\mathcal{B} = \{x, a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n\}$ is a basis of $\mathcal{X}$.

2. In R^3 let $x_1 = (2, 4, 6)$, $x_2 = (1, 5, 7)$ and $x_3 = (-2, 0, -2)$. Show that the set $\mathcal{A} = \{x_1, x_2, x_3\}$ is a basis and specify α_1, α_2 and α_3 so that

$$x = (10, -9, 1) = \sum_{i=1}^{3} \alpha_i x_i$$

3. Let P be the set of all polynomials over the reals (a typical element of P is of the form $p = p_0 + p_1 t + p_2 t^2 + \dots + p_n t^n$)
Demonstrate (exhibit) a basis for P and prove that it is a basis. What is the dimension of P?

4. On the real line, show that the intersection of the collection of all intervals of the form $I_n = (0, 1 + \frac{1}{n})$ for $n = 1, 2, \dots$ is not open. (Use the absolute value norm.)

5. Let F be a Banach Space. Show that the ball $B = \{ f \in F \mid \|f\|^2 \leq \alpha \}$ where α is a positive real number is convex.

1. 5
2. 12
3. 15
4. 15
5. 20

67%

1. $\mathcal{a} = \{a_1, a_2, \ldots a_n\}$

$x \in \mathfrak{X} \quad x \neq \theta$

$\forall x \in \mathfrak{X} \ \exists \ \{\alpha_i\}, \ i = 1, 2, \ldots, n$

$\ni x = \sum_{i=1}^{n} \alpha_i a_i$

if $x \neq \theta$, at least one $\alpha_i \neq 0$. Let one such α_i be α_K. Then

$$x = \sum_{\substack{i=1 \\ i \neq K}}^{n} \alpha_i a_i + \alpha_K a_K$$

$$\Rightarrow a_K = \frac{1}{\alpha_K}\left[x - \sum_{\substack{i=1 \\ i \neq K}}^{n} \alpha_i a_i\right]$$

Thus, since a_K is linearly dependent on x, we can substitute x for a_K in the basis set.

2. $x_1 = (2, 4, 6)$

$x_2 = (1, 5, 7)$

$x_3 = (-2, 0, -2)$

Dimension of R^3 is 3.

$\exists$ no $\alpha\beta \ni x_1 = \alpha x_2 + \beta x_3$, ~~$x_1 = \alpha x_3$; OR $x_2 = \alpha x_3$~~

$\Rightarrow x_1, x_2, x_3$ is a basis

LET

$x = (10, -9, 1)$

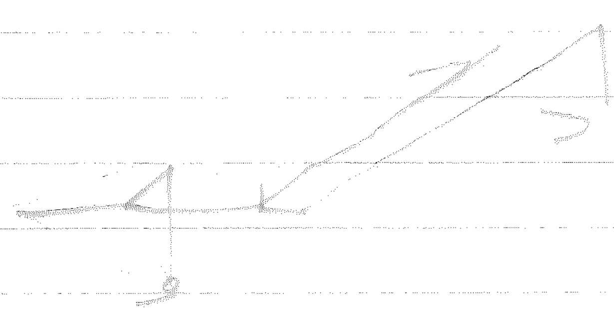
$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = (10, -9, 1)$

$\alpha_1 2 + \alpha_2 - 2\alpha_3 = 10$

$\vdots$

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & 0 \\ 6 & 7 & -2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -9 \\ 1 \end{bmatrix}$$

Solve for $\alpha_1, \alpha_2, \;\&\; \alpha_3$



3. A basis for P is

$\{t^m\}$; $m = 0, 1, 2, 3, 4, 5, \ldots$

$\forall\, p \in P$, $\exists\, \{p_m\}$, $m = 0, 1, 2, \ldots$

$\ni$

$$p = \sum_{m=0}^{\infty ?} p_m x^m$$

Note: [Some p_m might be zero]

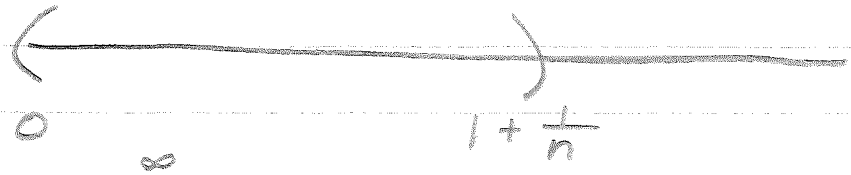
~~P is infinitely~~

P has infinite dimension since the polynomial order is not specified and can thus be arbitrarily large.

~~Other sets, like:~~

Most any polynomial set [illegible] Legendre, or Hermite, are also bases of P.

4. $I_n = (0,\ 1 + \frac{1}{n})$



$P = \bigcap_{n=1}^{\infty} I_n \quad ; n = 1, 2, 3, 4 \ldots$

clearly $P = (0,\ 1]$ since there does not exist an $\epsilon > 0$ $\ni$ the point $1+\epsilon$ can be excluded from P by choosing n sufficiently large. But $1 \in I_n$.

$\epsilon < 2$

ie, $\forall\ \epsilon > 0\ \exists\ N \ni 1 + \frac{1}{N} < 1 + \epsilon$.

$\therefore\ 1+\epsilon \notin P\ \forall\ \epsilon > 0$.

BUT $1 \in I_n\ \forall\ n \Rightarrow P = (0, 1]$

$(0, 1]$ open?

5. $F \Rightarrow$ BANACH SPACE

$B = \{ f \in F \mid \|f\|^2 \leq \alpha \}$

$\alpha > 0$

B is convex if for all γ $\ni$ $(0 \leq \gamma \leq 1)$ the point $\gamma f_1 + (1-\gamma) f_2 \in B \; \forall \; f_1, f_2 \in F$

$f_i \in F \Rightarrow \|f_i\|^2 \leq \alpha \quad ; i = 1, 2$

NOW:

$$\|\gamma f_1 + (1-\gamma) f_2\|$$
$$\leq \|\gamma f_1\| + \|(1-\gamma) f_2\|$$
$$= \gamma\|f_1\| + (1-\gamma)\|f_2\|$$
$$\leq \gamma\sqrt{\alpha} + (1-\gamma)\|f_2\|$$
$$\leq \gamma\sqrt{\alpha} + (1-\gamma)\sqrt{\alpha}$$
$$= \sqrt{\alpha}$$

or

$$\|\gamma f_1 + (1-\gamma) f_2\| \leq \alpha$$

$\therefore \gamma f_1 + (1-\gamma) f_2 \in F$ and

B is convex.

EE 5327 QUIZ 3 Fall 1977
Dr. Liberty

1. Given the set of functions $\{e^t, e^{2t}, e^{3t}\}$ on the real line interval $[0, 1]$, show that the set is linearly independent, and apply the Gram-Schmidt procedure to generate a new set $\{e_1(t), e_2(t), e_3(t)\}$ where each $e_i(t)$ is orthonormal in $L_2[0,1]$.

2. Given $x(t) = \sin 2\pi t$ find the $\hat{x} \in L_2[0,1]$ as a linear combination of the e_i's in problem 1 that is a best approximation to x in the $L_2[0,1]$ norm.

3. Consider all vectors x in R^3 satisfying

$$(x|y_1) = 1$$
$$(x|y_2) = 2$$

with $y_1 = (1, 2, 1)$ and $y_2 = (2, 1, 2)$
where $(x|y) = x^T Q y$ with

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix}.$$

Find the x satisfying these constraints with minimum norm.

BOB MARKS

1 25
1 24
2 24
3 25

98

QUIZ # 3 SOLUTIONS

1. SHOWING THE SET $\{e^t, e^{2t}, e^{3t}\}$ ON $[0, 1]$ ARE LINEARLY INDEPENDENT.

FROM THEOREM 1 ON PG. 20 OF LUENBERGER, WE SIMPLY NEED TO SHOW THAT

$$\Sigma_{k=1}^{3} \alpha_k e^{kt} = 0 \Rightarrow \alpha_k = 0, \ k = 1, 2, 3$$

NOW, FOR $X = e^t$:

$$\Sigma_k \alpha_k e^{kt} = \alpha_1 e^t + \alpha_2 e^{2t} + \alpha_3 e^{3t}; \ t \in [0, 1]$$

$$= \alpha_1 X + \alpha_2 X^2 + \alpha_3 X^3 \quad ; X \in [1, e]$$

FROM THE THEORY OF POLYNOMIALS, THIS SUM IS ZERO ON ANY CLOSED INTERVAL IFF $\alpha_k = 0$, $k = 1, 2, 3$.

APPLYING GRAM-SCHMIDT PROCEDURE, WE HAVE, FROM PG 54:

$$e_1(t) = e^t / \| e^t \|$$

NOW $\int_0^1 e^{kt} dt = \frac{1}{k} [e^k - 1]$

$$\Rightarrow \| e^{kt} \| = \sqrt{\frac{e^k - 1}{k}}$$

THUS

$$\Rightarrow \quad e_1(t) = \frac{e^t}{\sqrt{e - 1}} = 0.7629 \ e^t$$

FROM PG 55:

$$z_2(t) = e^{2t} - (e^{2t} | e_1) e_1$$

$$= e^{2t} - \frac{e^t}{e - 1} \int_0^1 e^{3t} dt$$

$$= e^{2t} - \frac{1}{3} \frac{e^3 - 1}{e - 1} e^t$$

$$= e^{2t} - 3.7024 \ e^t$$

$$\|z_2\|^2 = \int_0^1 [e^{4t} - 7.4048\, e^{3t} + 13.7081 e^{2t}]\, dt$$
$$= \frac{1}{4}(e^4 - 1) - \frac{7.4048}{3}(e^3 - 1) + \frac{13.7081}{2}(e^2 - 1)$$
$$= 10.0823 \Rightarrow \|z_2\| = 3.1753$$

$$\Rightarrow \quad e_2(t) = \frac{z_2(t)}{\|z_2\|} = 0.31494\, e^{2t} - 1.1660\, e^t$$

$$z_3(t) = e^{3t} - (e^{3t} | e_2)\, e_2 - (e^{3t} | e_1)\, e_1$$
$$= e^{3t} - \int_0^1 e^{3t} e_2(t)\,dt\; e_2(t)$$
$$- \int_0^1 e^{3t} e_1(t)\,dt\; e_1(t)$$
$$= e^{3t} - [0.31494\, e^{2t} - 1.1660\, e^t]$$
$$\times [0.31494 \int_0^1 e^{5t}\,dt - 1.1660 \int_0^1 e^{4t}\,dt]$$
$$- (0.7629)^2\, e^t \int_0^1 e^{4t}\,dt$$
$$= e^{3t} - [0.31494\, e^{2t} - 1.1660\, e^t]$$
$$\times \left[\frac{0.31494}{5}(e^5 - 1) - \frac{1.1660}{4}(e^4 - 1)\right]$$
$$- \frac{(0.7629)^2}{4}(e^4 - 1)\, e^t$$
$$= e^{3t} + [0.31494\, e^{2t} - 1.1660\, e^t]\; 6.3386$$
$$- 7.7988\, e^t$$
$$= e^{3t} + 1.9963\, e^{2t} - 15.190\, e^t$$
$$\|z_3\|^2 = \int_0^1 e^{6t} + 2(1.9963) \int_0^1 e^{5t}\,dt$$
$$+ [(1.9963)^2 - 2(15.190)] \int_0^1 e^{4t}\,dt$$
$$- 2(15.190)(1.9963) \int_0^1 e^{3t}\,dt + (15.190)^2 \int_0^1 e^{2t}\,dt$$
$$= \frac{1}{6}(e^6 - 1) + 2(1.9963)\frac{1}{5}(e^5 - 1)$$
$$+ [(1.9963)^2 - 2(15.190)]\frac{1}{4}(e^4 - 1)$$
$$- \frac{2}{3}(15.190)(1.9963)(e^3 - 1) + \frac{1}{2}(15.190)^2(e^2 - 1)$$
$$= 182.368 \Rightarrow \|z_3\| = 13.504$$
$$\Rightarrow e_3(t) = 0.07405\, e^{3t}$$
$$+ 0.1478\, e^{2t}$$
$$- 1.124\, e^t$$

IN SUMMARY:

$e_1(t) = 0.763\ e^{t}$

$e_2(t) = 0.315 \cdot e^{2t} - 1.17\, e^{t}$

$e_3(t) = 0.0741\, e^{3t} + 0.148\, e^{2t} - 1.12\, e^{t}$

2. $x(t) = \sin 2\pi t \quad ; \quad 0 \le t \le 1$

BY PROJECTION THEOREM:

$$\hat{x}(t) = (x|e_1)e_1 + (x|e_2)e_2 + (x|e_3)e_3$$

NOW

$$\int_0^1 e^{kt} \sin 2\pi t \, dt = \frac{e^{kt}}{k^2+(2\pi)^2}\left[k \sin 2\pi t - 2\pi \cos 2\pi t\right]_0^1$$

$$= \frac{1}{k^2+(2\pi)^2}\left[-2\pi e^k + 2\pi\right]$$

$$= \frac{2\pi}{k^2+(2\pi)^2}\left[1 - e^k\right] \quad < 0$$

THUS

$$(x|e_1) = 0.7629 \int_0^1 e^t \sin 2\pi t \, dt$$

$$= -0.7629 \frac{2\pi}{1+(2\pi)^2}\left[e-1\right] =$$

$$= -0.20347$$

$$(x|e_2) = 0.31494 \int_0^1 e^{2t} \sin 2\pi t \, dt - 1.1660 \int_0^1 e^t \sin 2\pi t \, dt$$

$$= -0.31494 \frac{2\pi}{1+(2\pi)^2}\left[e-1\right] + 1.1660 \frac{2\pi}{4+(2\pi)^2}\left[e^2-1\right]$$

$$= -0.08400 + 1.07657$$

$$= 0.9926$$

$$(x|e_3) = -0.07405 \frac{2\pi}{9+(2\pi)^2}(e^3-1) - 0.1478 \frac{2\pi}{4+(2\pi)^2}(e^2-1) + 1.124 \frac{2\pi}{1+(2\pi)^2}(e-1)$$

$$= -0.1832 - 0.1365 + 0.5851$$

$$= 0.2654$$

IN SUMMARY:

$$\hat{x}(t) = -0.203\, e_1(t) + 0.993\, e_2(t) + 0.265\, e_3(t) \quad ; \quad 0 \le t \le 1$$

3. $(x|y_1) = 1$, $(x|y_2) = 2$

$y_1^T = (1, 2, 1)$ $y_2^T = (2, 1, 2)$

$(x|y) = x^T Q y$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

FINDING THE X OF MINIMUM NORM, X_0, SATISFYING THESE CONSTRAINTS IS DONE BY STRAIGHTFORWARD APPLICATION OF THEOREM 2 ON PAGE 65:

$$X_0 = \beta_1 y_1 + \beta_2 y_2$$

WHERE

$$(y_1|y_1)\beta_1 + (y_2|y_1)\beta_2 = C_1 = 1$$
$$(y_1|y_2)\beta_1 + (y_2|y_2)\beta_2 = C_2 = 2$$

OR:

$$\begin{bmatrix} (y_1|y_1) & (y_2|y_1) \\ (y_1|y_2) & (y_2|y_2) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

NOW, TO FIND THE INNER PRODUCTS:

$$(y_1|y_1) = [1 \quad 2 \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= [1 \quad 2 \quad 1] \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

$$= 26$$

$$(y_2|y_1) = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

$$= 31$$

$$(y_1|y_2) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 13 \end{bmatrix}$$

$$= 31 = (y_2|y_1) \quad \text{(AS IT SHOULD)}$$

$$(y_2|y_2) = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 13 \end{bmatrix}$$

$$= 38$$

THUS:

$$\begin{bmatrix} 26 & 31 \\ 31 & 38 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

SOLVE BY CRAMER'S RULE:

$$\det \begin{bmatrix} 26 & 31 \\ 31 & 38 \end{bmatrix} = 27$$

$$\Delta_1 = \det \begin{bmatrix} 1 & 31 \\ 2 & 38 \end{bmatrix} = -24$$

$$\Rightarrow \beta_1 = -24/27 = -8/9$$

$$\Delta_2 = \det \begin{bmatrix} 26 & 1 \\ 31 & 2 \end{bmatrix} = 21$$

$$\Rightarrow \beta_2 = 21/27 = 7/9$$

THEREFORE:

$$\begin{aligned} x_0 &= \beta_1 y_1 + \beta_2 y_2 \\ &= -\tfrac{8}{9}\left[1, 2, 1\right]^T + \tfrac{7}{9}\left[2, 1, 2\right] \\ &= \left[-\tfrac{8}{9}, -\tfrac{16}{9}, -\tfrac{8}{9}\right]^T + \left[\tfrac{14}{9}, \tfrac{7}{9}, \tfrac{14}{9}\right]^T \\ &= \left[\tfrac{6}{9}, -\tfrac{9}{9}, \tfrac{6}{9}\right]^T \\ &= \left[\tfrac{2}{3}, -1, \tfrac{2}{3}\right]^T \end{aligned}$$

EE 5327 Take Home Exam Fall 1977

Instructions: You may not communicate with anyone concerning this examination except Dr. Liberty. You may use reference material if necessary but you must list the specific material you utilize. Keep in mind that your objective should be to accept the challenge of the exam and conquer it with a minimum of textual aid.

Problems 1 - 4. Do problems 1 - 4 on pg. 209 of Luenberger.

Problems 5. Let P be the space of all polynomial functions over the reals of degree three or less. Let L be a linear transformation from P into P defined by

$$L \triangleq [D^2 + 5D + 6]$$

where $D \equiv \frac{d}{dx}$ and a typical element of P is of the form

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3.$$

Let $\mathcal{B}$ be the ordered basis for P consisting of the functions $f_i(x) = x^{i-1}$ for $i = 1, 2, 3, 4$. Find the matrix representation of L in the ordered basis $\mathcal{B}$.

Problem 6. For the same space as in problem 5 let L = D.

(i) Find the matrix representation, $[L]_{\mathcal{B}}$, of L in the ordered basis $\mathcal{B}$.

(ii) Let t be a real number and define $g_i(x) = (x + t)^{i-1}$, $i = 1,2,3,4$. Show that the set $A = \{g_1, g_2, g_3, g_4\}$ is a basis for P and find the matrix U such that $U^{-1}[L]_{\mathcal{B}}U$ is the matrix representation of L in the ordered basis A.

EE 5327 In Class Exam Fall 1977

Dr. Liberty

1. Regarding our discussion over the validity of the proof of Theorem 2 (Riesz-Fréchet) on pg. 109, let f be a nonzero bounded linear functional on a Hilbert space H. Let $N = \{x : f(x) = 0\}$ and let $z \in N^{\perp}$ such that $f(z) = 1$. Show directly (don't use Thm. 2 or its proof) that z is unique. What is the dimension of the subspace $N^{\perp}$?

2. Let M be a subspace of a Hilbert space H and let $f : H \to R$ be defined by $f(x) = f(\alpha x_1 + m) = \alpha$ where $x_1 \in M^{\perp}$ and $m \in M$. Is f linear? Is f bounded? (Prove)

Extra Credit

3. Attempt (but only after doing all you can on problems 1 & 2) to formulate and solve problem 6 of chapter 5. Try to write the problem as a minimum norm problem in $L_\infty[0,T]$ by integrating the equations of motion over the interval $[0,T]$